ON THE RESTRICTION OF ZUCKERMAN'S DERIVED FUNCTOR MODULES $A_{\mathfrak{q}}(\lambda)$ TO REDUCTIVE SUBGROUPS

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Abstract. In this paper, we study the restriction of Zuckerman's derived functor $(\mathfrak{g}, K)$-modules $A_{\mathfrak{q}}(\lambda)$ to $\mathfrak{g}'$ for symmetric pairs of reductive Lie algebras $(\mathfrak{g}, \mathfrak{g}')$. When the restriction decomposes into irreducible $(\mathfrak{g}', K')$-modules, we give an upper bound for the branching law. In particular, we prove that each $(\mathfrak{g}', K')$-module occurring in the restriction is isomorphic to a submodule of $A_{\mathfrak{q}'}(\lambda')$ for a parabolic subalgebra $\mathfrak{q}'$ of $\mathfrak{g}'$, and determine their associated varieties. For the proof, we construct $A_{\mathfrak{q}}(\lambda)$-modules on complex partial flag varieties by using $\mathcal{D}$-modules.

1. Introduction

Our object of study is branching laws of Zuckerman's derived functor modules $A_{\mathfrak{q}}(\lambda)$ with respect to symmetric pairs of real reductive Lie groups.

Let G_0 be a real reductive Lie group with Lie algebra $\mathfrak{g}_0$. Fix a Cartan involution θ of G_0 so that the fixed set $K_0 := (G_0)^\theta$ is a maximal compact subgroup of G_0. Write K for the complexification of K_0 and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ for the Cartan decomposition with respect to θ. The cohomologically induced module $A_{\mathfrak{q}}(\lambda)$ is a $(\mathfrak{g}, K)$-module defined for a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g} := \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ and a character λ. The $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ is unitarizable under a certain condition on the parameter λ and therefore plays a large part in the study of the unitary dual of real reductive Lie groups.

One of the fundamental problems in the representation theory is to decompose a given representation into irreducible constituents. To begin with, we consider the restriction of $(\mathfrak{g}, K)$-modules to K, or equivalently, to the compact group K_0. In this case, any irreducible $(\mathfrak{g}, K)$-module decomposes as the direct sum of irreducible representations of K and each K-type occurs with finite multiplicity. For $A_{\mathfrak{q}}(\lambda)$-modules, the following formula gives an upper bound for the multiplicities.

Fact 1.1 ([8, §V.4])**.** *Let $\mathfrak{u}$ be the nilradical of $\mathfrak{q}$. Take a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$ such that $\mathfrak{t} \subset \mathfrak{q} \cap \mathfrak{k}$ and choose a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ contained in $\Delta(\mathfrak{q} \cap \mathfrak{k}, \mathfrak{t})$. For a dominant integral weight $\mu \in \mathfrak{t}^*$ write $F(\mu)$ for the irreducible finite-dimensional representation of K with highest weight μ. Then*

$$A_{\mathfrak{q}}(\lambda)|_K \leq \bigoplus_{p=0}^{\infty} \bigoplus_{\mu} F(\mu)^{\oplus m(\mu, p)}, \tag{1.1}$$

where $m(\mu, p)$ is the multiplicity of weight μ in $\mathbb{C}_{\lambda + 2\rho(\mathfrak{u} \cap \mathfrak{p})} \otimes S^p(\mathfrak{u} \cap \mathfrak{p})$.

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There is also an explicit branching formula of $A_{\mathfrak{q}}(\lambda)|_K$ for weakly fair λ, known as the generalized Blattner formula (see [1, §II.7], [8, §V.5]).

On the other hand, the restriction to a non-compact subgroup is more complicated. Let σ be an involution of G_0 that commutes with θ and let G'_0 be the identity component of $(G_0)^\sigma$. The pair (G_0, G'_0) is called a symmetric pair. Write $\mathfrak{g}'$ for the complexified Lie algebra of G'_0 and write K' for the complexification of the maximal compact group $K'_0 := (G'_0)^\theta$ of G'_0. If G'_0 is non-compact, the restriction $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}',K')}$ does not decompose into irreducible $(\mathfrak{g}', K')$-modules in general. Indeed, $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}',K')}$ does not have any irreducible submodule in many cases.

Nevertheless, there are classes of $(\mathfrak{g}, K)$-modules which decompose into irreducible $(\mathfrak{g}', K')$-modules and explicit branching formulas were obtained for some particular representations [3], [4], [9], [10], [14], [16], [19], [20]. In his series of papers [9], [10], [11], [12], Kobayashi introduced the notion of discretely decomposable $(\mathfrak{g}', K')$-modules and gave criteria for the discretely decomposable restrictions (see Fact 5.5). By virtue of this result, we can single out $A_{\mathfrak{q}}(\lambda)$-modules that decompose into irreducible $(\mathfrak{g}', K')$-modules. See [15] for a classification of the discretely decomposable restrictions $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}',K')}$. Recent developments on these subjects are discussed in [13].

Our aim is to find a branching law of $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}',K')}$ when it is discretely decomposable. The main results of this paper are Theorem 6.3 and its reformulation Theorem 6.4, where we construct an injective $(\mathfrak{g}', K')$-homomorphism:

$$A_{\mathfrak{q}}(\lambda) \to \bigoplus_{p=0}^{\infty} \bigoplus_{\lambda'} A_{\mathfrak{q}''}(\lambda')^{\oplus m(\lambda', p)}. \tag{1.2}$$

The parabolic subalgebra $\mathfrak{q}''$ of $\mathfrak{g}'$ and the multiplicity function $m(\lambda', p)$ are given in (5.1) and (6.6), respectively. Theorem 6.4 is a generalization of Fact 1.1 because if $\theta = \sigma$, then $G'_0 = K_0$ and it turns out that the right side of (1.2) is isomorphic to the right side of (1.1) as a K-module.

For the proof of these theorems, we realize $A_{\mathfrak{q}}(\lambda)$-modules as the global sections of sheaves on complex partial flag varieties in Theorem 4.1, using $\mathcal{D}$-modules. A relation between cohomologically induced modules and twisted $\mathcal{D}$-modules on the complete flag variety was constructed by Hecht–Miličić–Schmid–Wolf [5]. See [1], [7], [17] for further developments of this result. Our proof of Theorem 4.1 is based on [5].

This paper is organized as follows. In Section 2, we recall the definitions of cohomological induction and $A_{\mathfrak{q}}(\lambda)$-modules, following the book by Knapp–Vogan [8]. In this paper, we extend actions of a compact group K_0 to actions of its complexification K, and view $(\mathfrak{g}, K_0)$-modules as $(\mathfrak{g}, K)$-modules. In Section 3, we fix notation and prove lemmas concerning homogeneous spaces and differential operators. Lemma 3.4 is used in the proof of Theorem 4.1. Section 4 is devoted to the proof of Theorem 4.1. In Section 5, we construct θ-stable parabolic subalgebras of $\mathfrak{g}'$ that will appear in the branching laws, using a criterion for the discrete decomposability given in [12]. The parabolic subalgebra $\mathfrak{q}'$ is defined in Theorem 5.4 and $\mathfrak{q}''$ is defined in (5.1). We prove Theorem 6.3 and Theorem 6.4 in Section 6. We study the associated varieties of $(\mathfrak{g}, K)$-modules in Section 7. As a corollary to Theorem 6.4, we determine the associated variety of the irreducible constituents of $A_{\mathfrak{q}}(\lambda)|_{\mathfrak{g}'}$ in Theorem 7.5.

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2. Cohomological Induction

In this section, we fix notation concerning cohomological induction and $A_{\mathfrak{q}}(\lambda)$-modules, following [8].

Let K_0 be a compact Lie group. The complexification K of K_0 has the structure of reductive linear algebraic group. Since any locally finite action of K_0 is uniquely extended to an algebraic action of K, the locally finite K_0-modules are identified with the algebraic K-modules.

Define the Hecke algebra $R(K_0)$ as the space of K_0-finite distributions on K_0. For $S \in R(K_0)$, the pairing with a smooth function $f \in C(K_0)$ on K_0 is written as

$$\int_{K_0} f(k) dS(k).$$

The product of $S, T \in R(K_0)$ is given by

$$S * T : f \mapsto \int_{K_0 \times K_0} f(kk') dS(k) dT(k').$$

The associative algebra $R(K_0)$ does not have the identity, but has an approximate identity (see [8, Chapter I]). The locally finite K_0-modules are identified with the approximately unital left $R(K_0)$-modules. The action map $R(K_0) \times V \to V$ is given by

$$(S, v) \mapsto \int_{K_0} kv \, dS(k)$$

for a locally finite K_0-module V. Here, kv is regarded as a smooth function on K_0 that takes values on V. If dk_0 denotes the Haar measure of K_0, then $R(K_0)$ is identified with the K-finite smooth functions $C(K_0)_{K_0}$ by $f dk_0 \mapsto f$ and hence with the regular functions $\mathcal{O}(K)$ on K. As a $\mathbb{C}$-algebra, we have a canonical isomorphism

$$R(K_0) \simeq \bigoplus_{\tau \in \widehat{K}} \mathrm{End}(V_\tau),$$

where $\widehat{K}$ is the set of equivalence classes of irreducible K-modules, and V_τ is a representation space of $\tau \in \widehat{K}$. Hence $R(K_0)$ depends only on the complexification K, so in what follows, we also denote $R(K_0)$ by $R(K)$.

The Hecke algebra $R(K)$ is generalized to $R(\mathfrak{g}, K)$ for the following pairs $(\mathfrak{g}, K)$.

Definition 2.1. Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra and let K be a complex reductive linear algebraic group with Lie algebra $\mathfrak{k}$. Suppose that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and that an algebraic group homomorphism $\phi : K \to \mathrm{Aut}(\mathfrak{g})$ is given. We say that $(\mathfrak{g}, K)$ is a *pair* if the following two assumptions hold.

- The restriction $\phi(k)|_{\mathfrak{k}}$ is equal to the adjoint action $\mathrm{Ad}(k)$ for $k \in K$.
- The differential of ϕ is equal to the adjoint action $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$.

Remark 2.2. Let G be a complex algebraic group and K a reductive linear algebraic subgroup. Then the Lie algebra $\mathfrak{g}$ of G and K form a pair with respect to the adjoint action $\phi(k) := \mathrm{Ad}(k)$ for $k \in K$. All the pairs we will consider in the following are given in this way.

Definition 2.3. For a pair $(\mathfrak{g}, K)$, let V be a complex vector space with a Lie algebra action of $\mathfrak{g}$ and an algebraic action of K. We say that V is a $(\mathfrak{g}, K)$-*module* if

- the differential of the action of K coincides with the restriction of the action of $\mathfrak{g}$ to $\mathfrak{k}$; and
- $(\phi(k)\xi)v = k(\xi(k^{-1}(v)))$ for $k \in K$, $\xi \in \mathfrak{g}$, and $v \in V$.

We write $\mathcal{C}(\mathfrak{g}, K)$ for the category of $(\mathfrak{g}, K)$-modules.

Let $(\mathfrak{g}, K)$ be a pair in the sense of Definition 2.1. We extend the representation $\phi : K \to \mathrm{Aut}(\mathfrak{g})$ to a representation on the universal enveloping algebra $\phi : K \to \mathrm{Aut}(U(\mathfrak{g}))$. Define the Hecke algebra $R(\mathfrak{g}, K)$ as

$$R(\mathfrak{g}, K) := R(K) \otimes_{U(\mathfrak{k})} U(\mathfrak{g}).$$

The product is given by

$$(S \otimes \xi) \cdot (T \otimes \eta) = \sum_i (S * (\langle \xi_i^*, \phi(\cdot)^{-1}\xi \rangle T) \otimes \xi_i \eta)$$

for $S, T \in R(K)$ and $\xi, \eta \in U(\mathfrak{g})$. Here ξ_i is a basis of the linear span of $\phi(K)\xi$ and ξ^i is its dual basis. As in the group case, the $(\mathfrak{g}, K)$-modules are identified with the approximately unital left $R(\mathfrak{g}, K)$-modules. The action map $R(\mathfrak{g}, K) \times V \to V$ is given by

$$(S \otimes \xi,\, v) \mapsto \int_{K_0} k(\xi v)\, dS(k)$$

for a $(\mathfrak{g}, K)$-module V.

Let $(\mathfrak{g}, K)$ and $(\mathfrak{h}, M)$ be pairs in the sense of Definition 2.1. Let $i : (\mathfrak{h}, M) \to (\mathfrak{g}, K)$ be a map between pairs, namely, a Lie algebra homomorphism $i_{\mathrm{alg}} : \mathfrak{h} \to \mathfrak{g}$ and an algebraic group homomorphism $i_{\mathrm{gp}} : M \to K$ satisfy the following two assumptions.

- The restriction of i_{alg} to the Lie algebra $\mathfrak{m}$ of M is equal to the differential of i_{gp}.
- $\phi_K(m) \circ i_{\mathrm{alg}} = i_{\mathrm{alg}} \circ \phi_M(m)$ for $m \in M$, where ϕ_K denotes ϕ for $(\mathfrak{g}, K)$ in Definition 2.1 and ϕ_M denotes ϕ for $(\mathfrak{h}, M)$.

We define covariant functors $P_{\mathfrak{h},M}^{\mathfrak{g},K} : \mathcal{C}(\mathfrak{h}, M) \to \mathcal{C}(\mathfrak{g}, K)$ and $I_{\mathfrak{h},M}^{\mathfrak{g},K} : \mathcal{C}(\mathfrak{h}, M) \to \mathcal{C}(\mathfrak{g}, K)$ as

$$P_{\mathfrak{h},M}^{\mathfrak{g},K} : V \mapsto R(\mathfrak{g}, K) \otimes_{R(\mathfrak{h},M)} V,$$
$$I_{\mathfrak{h},M}^{\mathfrak{g},K} : V \mapsto (\mathrm{Hom}_{R(\mathfrak{h},M)}(R(\mathfrak{g}, K),\, V))_K,$$

where $(\cdot)_K$ is the subspace of K-finite vectors. Then $P_{\mathfrak{h},M}^{\mathfrak{g},K}$ is right exact and $I_{\mathfrak{h},M}^{\mathfrak{g},K}$ is left exact. Write $(P_{\mathfrak{h},M}^{\mathfrak{g},K})_j$ for the j-th left derived functor of $P_{\mathfrak{h},M}^{\mathfrak{g},K}$ and write $(I_{\mathfrak{h},M}^{\mathfrak{g},K})^j$ for the j-th right derived functor of $I_{\mathfrak{h},M}^{\mathfrak{g},K}$.

In the context of unitary representations of real reductive Lie groups, we are especially interested in the $(\mathfrak{g}, K)$-modules cohomologically induced from one-dimensional representations of a certain type of parabolic subalgebras, which are called $A_{\mathfrak{q}}(\lambda)$-modules.

Let G_0 be a connected real linear reductive Lie group with Lie algebra $\mathfrak{g}_0$. This means that G_0 is a connected closed subgroup of $GL(n, \mathbb{R})$ and stable under transpose. We fix such an embedding and write G for the connected algebraic subgroup

of $GL(n, \mathbb{C})$ with Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \sqrt{-1}\mathfrak{g}_0$. In what follows, we embed reductive subgroups of G_0 in $GL(n, \mathbb{C})$ and define their complexifications similarly.

Fix a Cartan involution θ so the θ-fixed point set $K_0 = G_0^\theta$ is a maximal compact subgroup of G_0. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the corresponding Cartan decomposition. We let θ also denote the induced involution on $\mathfrak{g}_0$ and its complex linear extension to $\mathfrak{g}$.

Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ that is stable under θ. The normalizer $N_{G_0}(\mathfrak{q})$ of $\mathfrak{q}$ in G_0 is denoted by L_0. The complexified Lie algebra $\mathfrak{l}$ of L_0 is a Levi part of $\mathfrak{q}$. Let bar $x \mapsto \bar{x}$ denote the complex conjugate with respect to the real form $\mathfrak{g}_0$. Then we have $\mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{l}$ and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ for the nilradical $\mathfrak{u}$ of $\mathfrak{q}$.

Because $L \cap K$ is connected, one-dimensional $(\mathfrak{l}, L \cap K)$-modules are determined by the action of the center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$. Let $\mathbb{C}_\lambda$ denote the one-dimensional $(\mathfrak{l}, L \cap K)$-module corresponding to $\lambda \in \mathfrak{z}(\mathfrak{l})^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{z}(\mathfrak{l}), \mathbb{C})$. With our normalization, the trivial representation corresponds to $\mathbb{C}_0$. The top exterior product $\bigwedge^{\mathrm{top}}(\mathfrak{g}/\bar{\mathfrak{q}})$ regarded as an $(\mathfrak{l}, L \cap K)$-module by the adjoint action corresponds to $\mathbb{C}_{2\rho(\mathfrak{u})}$ for $2\rho(\mathfrak{u}) := \mathrm{Trace}\,\mathrm{ad}_{\mathfrak{u}}(\cdot)$.

Definition 2.4. Let $\mathbb{C}_\lambda$ be a one-dimensional $(\mathfrak{l}, L \cap K)$-module.

We say λ is *unitary* if λ takes pure imaginary values on the center $\mathfrak{z}(\mathfrak{l}_0)$ of $\mathfrak{l}_0$, or equivalently, if $\mathbb{C}_\lambda$ is the underlying $(\mathfrak{l}, L \cap K)$-module of a unitary character of L_0.

We say λ is *linear* if $\mathbb{C}_\lambda$ lifts to an algebraic representation of the complexification L of L_0.

Remark 2.5. If λ is linear, then λ takes real values on $\mathfrak{z}(\mathfrak{l}_0) \cap \mathfrak{p}_0$. In particular, if λ is linear and unitary, then λ is zero on $\mathfrak{z}(\mathfrak{l}) \cap \mathfrak{p}$.

Let $\mathbb{C}_\lambda$ be a one-dimensional $(\mathfrak{l}, L \cap K)$-module. We see $\mathbb{C}_{\lambda+2\rho(\mathfrak{u})} \simeq \mathbb{C}_\lambda \otimes \mathbb{C}_{2\rho(\mathfrak{u})}$ as a $(\bar{\mathfrak{q}}, L \cap K)$-module (resp. a $(\mathfrak{q}, L \cap K)$-module) by letting $\bar{\mathfrak{u}}$ (resp. $\mathfrak{u}$) acts as zero. Then, for inclusion maps of pairs $(\bar{\mathfrak{q}}, L \cap K) \to (\mathfrak{g}, K)$ and $(\mathfrak{q}, L \cap K) \to (\mathfrak{g}, K)$, define the cohomologically induced modules $(P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_j(\mathbb{C}_{\lambda+2\rho(\mathfrak{u})})$ and $(I^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^j(\mathbb{C}_{\lambda+2\rho(\mathfrak{u})})$.

The functor $P^{\mathfrak{g},K}_{\mathfrak{g},L\cap K}$ is called the Bernstein functor and denoted by $\Pi^K_{L\cap K}$. Since $P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K} \simeq \Pi^K_{L\cap K} \circ P^{\mathfrak{g},L\cap K}_{\bar{\mathfrak{q}},L\cap K}$ and $P^{\mathfrak{g},L\cap K}_{\bar{\mathfrak{q}},L\cap K}$ is exact, it follows that $(P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_j \simeq (\Pi^K_{L\cap K})_j \circ P^{\mathfrak{g},L\cap K}_{\bar{\mathfrak{q}},L\cap K}$ for the j-th left derived functor $(\Pi^K_{L\cap K})_j$ of $\Pi^K_{L\cap K}$. Therefore, we have

$$(P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_j(\mathbb{C}_{\lambda+2\rho(\mathfrak{u})}) \simeq (\Pi^K_{L\cap K})_j(U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} \mathbb{C}_{\lambda+2\rho(\mathfrak{u})}).$$

Similarly, $\Gamma^K_{L\cap K} := I^{\mathfrak{g},K}_{\mathfrak{g},L\cap K}$ is called the Zuckerman functor and we have

$$(I^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^j(\mathbb{C}_{\lambda+2\rho(\mathfrak{u})}) \simeq (\Gamma^K_{L\cap K})^j(\mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), \mathbb{C}_{\lambda+2\rho(\mathfrak{u})})_{L\cap K})$$

for the j-th right derived functor $(\Gamma^K_{L\cap K})^j$ of $\Gamma^K_{L\cap K}$. Put $s = \dim(\mathfrak{u} \cap \mathfrak{k})$. We define

$$A_{\mathfrak{q}}(\lambda) := (P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_s(\mathbb{C}_{\lambda+2\rho(\mathfrak{u})}) \simeq (\Pi^K_{L\cap K})_s(U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} \mathbb{C}_{\lambda+2\rho(\mathfrak{u})}).$$

We now discuss the positivity of the parameter λ. Let $\mathfrak{h}_0$ be a fundamental Cartan subalgebra of $\mathfrak{l}_0$. Choose a positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ of the root system $\Delta(\mathfrak{g}, \mathfrak{h})$ such that $\Delta^+(\mathfrak{g}, \mathfrak{h}) \subset \Delta(\mathfrak{q}, \mathfrak{h})$ and put

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha.$$

We fix a non-degenerate invariant form $\langle\cdot,\cdot\rangle$ that is positive definite on the real span of the roots. In the following definition, we extend characters of $\mathfrak{z}(\mathfrak{l})$ to $\mathfrak{h}$ by zero on $[\mathfrak{l},\mathfrak{l}]\cap\mathfrak{h}$.

Definition 2.6. Let $\mathbb{C}_\lambda$ be a one-dimensional $(\mathfrak{l}, L\cap K)$-module. We say λ is in the *good range* (resp. *weakly good range*) if

$$\mathrm{Re}\,\langle\lambda+\rho(\mathfrak{n}),\alpha\rangle > 0 \text{ (resp. } \geq 0) \text{ for } \alpha\in\Delta(\mathfrak{u},\mathfrak{h}),$$

and in the *fair range* (resp. *weakly fair range*) if

$$\mathrm{Re}\,\langle\lambda+\rho(\mathfrak{u}),\alpha\rangle > 0 \text{ (resp. } \geq 0) \text{ for } \alpha\in\Delta(\mathfrak{u},\mathfrak{h}).$$

Definition 2.7. Let V be a $(\mathfrak{g},K)$-module. We say V is *unitarizable* if V admits a Hermitian inner product with respect to which $\mathfrak{g}_0$ acts by skew-Hermitian operators on V.

The $(\mathfrak{g},K)$-module $A_\mathfrak{q}(\lambda)$ has the following properties.

Fact 2.8 ([8])**.** *Let $\mathbb{C}_\lambda$ be a one-dimensional $(\mathfrak{l}, L\cap K)$-module.*

(i) *$A_\mathfrak{q}(\lambda)$ is of finite length as a $(\mathfrak{g},K)$-module.*
(ii) *If λ is in the weakly good range, $A_\mathfrak{q}(\lambda)$ is irreducible or zero.*
(iii) *If λ is in the good range, $A_\mathfrak{q}(\lambda)$ is nonzero.*
(iv) *If λ is unitary and in the weakly fair range, then $A_\mathfrak{q}(\lambda)$ is unitarizable.*

3. Differential Operators on Homogeneous Spaces

We introduce notation and lemmas concerning homogeneous spaces and differential operators, used in the subsequent sections. Let G be a complex linear algebraic group acting on a smooth variety X. Then the infinitesimal action is defined as a Lie algebra homomorphism from the Lie algebra $\mathfrak{g}$ of G to the space of vector fields $\mathcal{T}(X)$ on X. Denote the image of $\xi\in\mathfrak{g}$ by $\xi_X\in\mathcal{T}(X)$. Then ξ_X gives a first order differential operator on the structure sheaf $\mathcal{O}_X$.

Suppose that $X=G$ and the action of G on X is the product from left:

$$G\to\mathrm{Aut}(G),\qquad g\mapsto(g'\mapsto gg')$$

In this case we write the vector field ξ_X as ξ^L_G, which is a right invariant vector field on G. Similarly, if the action of G on $X=G$ is the product from right:

$$G\to\mathrm{Aut}(G),\qquad g\mapsto(g'\mapsto g'g^{-1}),$$

we write the vector field ξ_X as ξ^R_G, which is a left invariant vector field on G. Let $\xi_1,\cdots,\xi_n$ be a basis of $\mathfrak{g}$ and write $\xi^1,\cdots,\xi^n\in\mathfrak{g}^*$ for the dual basis. Define regular functions α^i_j,β^j_i on G for $1\leq i,j\leq n$ by

$$\alpha^i_j(g):=\langle\xi^i,\mathrm{Ad}(g^{-1})\xi_j\rangle,\quad \beta^j_i(g):=\langle\xi^j,\mathrm{Ad}(g)\xi_i\rangle. \tag{3.1}$$

Then it follows that

$$(\xi_j)^L_G=-\sum_{i=1}^n\alpha^i_j\cdot(\xi_i)^R_G,\quad (\xi_i)^R_G=-\sum_{j=1}^n\beta^j_i\cdot(\xi_j)^L_G,\quad \sum_{j=1}^n\alpha^i_j\beta^j_k=\delta^i_k.$$

We see $(\xi_j)^L_G$ as a differential operator on G. Then the function $(\xi_j)^L_G(\beta^j_i)$ on G is written as

$$(\xi_j)^L_G(\beta^j_i)=-\langle\xi^j,[\xi_j,\mathrm{Ad}(\cdot)\xi_i]\rangle.$$

Hence

$$\sum_{j=1}^{n}(\xi_j)^L_G(\beta_i^j) = -\sum_{j=1}^{n}\langle \xi^j, [\xi_j, \mathrm{Ad}(\cdot)\xi_i]\rangle = \operatorname{Trace}\operatorname{ad}(\mathrm{Ad}(\cdot)\xi_i). \tag{3.2}$$

Let H be a complex algebraic subgroup of G. The quotient $X := G/H$ is defined as a smooth algebraic variety (see [2, §II.6]). Denote by $\pi : G \to X$ the quotient map. Let V be a complex vector space with an algebraic action ρ of H. We define the *$\mathcal{O}_X$-module $\mathcal{V}_X$ associated with V* as the subsheaf of $\pi_*\mathcal{O}_G \otimes V$ given by

$$\mathcal{V}_X(U) := \{f \in \mathcal{O}(\pi^{-1}(U)) \otimes V : f(gh) = \rho(h)^{-1}f(g)\}$$

for an open set $U \subset X$. Here, we identify sections of $\mathcal{O}(\pi^{-1}(U)) \otimes V$ with regular V-valued functions on $\pi^{-1}(U)$. Analogous identification will be used for other varieties. The $\mathcal{O}_X$-module $\mathcal{V}_X$ corresponds to the G-equivariant vector bundle with typical fiber V.

The G-equivariant structure on $\mathcal{O}_G$ by the left translation induces a G-equivariant structure on $\mathcal{V}_X$. By differentiating it, the infinitesimal action of $\xi \in \mathfrak{g}$ is given by $f \mapsto \xi^L_G f$.

We write $\mathrm{Ind}_H^G(V)$ for the space of global sections $\Gamma(X, \mathcal{V}_X)$ regarded as an algebraic G-module. Then by the Frobenius reciprocity,

$$\mathrm{Hom}_G(W, \mathrm{Ind}_H^G(V)) \xrightarrow{\sim} \mathrm{Hom}_H(W, V)$$

for any algebraic G-module W.

Lemma 3.1. *If G and H are reductive, then*

$$R(G) \otimes_{R(H)} V \simeq \mathrm{Ind}_H^G(V)$$

as G-modules.

Proof. We give the H-action on $\mathcal{O}(G) \otimes_{\mathbb{C}} V$ by $h(f \otimes v) \mapsto f(\,\cdot\, h) \otimes hv$. The H-module $\mathcal{O}(G) \otimes_{\mathbb{C}} V$ decomposes as a direct sum of irreducible factors because H is reductive. From the definition of $\mathcal{V}_X$, the space of global sections $\mathrm{Ind}_H^G(V)$ is equal to the set of H-invariant elements $(\mathcal{O}(G) \otimes_{\mathbb{C}} V)^H$. With the identification $\mathcal{O}(G) \simeq R(G)$, we see that the canonical surjective map $R(G) \otimes_{\mathbb{C}} V \to R(G) \otimes_{R(H)} V$ is the projection onto the H-invariants. Hence we have

$$R(G) \otimes_{R(H)} V \simeq (\mathcal{O}(G) \otimes_{\mathbb{C}} V)^H \simeq \mathrm{Ind}_H^G(V)$$

as G-modules. □

Suppose that H' is another algebraic subgroup of G such that $H \subset H'$. Let $X' := G/H'$ and $S := H'/H$ be the quotient varieties and $\varpi : X \to X'$ the canonical map. Write $\mathcal{V}_S$ for the $\mathcal{O}_S$-module associated with V. Let $W := \mathrm{Ind}_H^{H'}(V)$ and let $\mathcal{W}_{X'}$ be the $\mathcal{O}_{X'}$-module associated with the H'-module W.

The following lemma is immediate from the definition, which indicates 'induction by stages' in our setting.

Lemma 3.2. *In the setting above, there is a canonical G-equivariant isomorphism $\varpi_*\mathcal{V}_X \to \mathcal{W}_{X'}$.*

Let K be an algebraic subgroup of G. The inclusion map $i : K \to G$ induces the immersion $i : Y := K/(H \cap K) \to X$ of algebraic variety. Define the ideal $\mathcal{I}_Y$ of $\mathcal{O}_X$ as

$$\mathcal{I}_Y := \{f \in \mathcal{O}_X : f(y) = 0 \text{ for } y \in Y\},$$

so $\mathcal{I}_Y$ is the defining ideal of the closure $\overline{Y}$ of Y. We denote by $\mathcal{I}_Y^p$ the p-th power of $\mathcal{I}_Y$ for $p \geq 0$. We use i^{-1} for the inverse image of sheaves of abelian groups. Then $i^{-1}(\mathcal{I}_Y^p/\mathcal{I}_Y^{p+1})$ is isomorphic to the K-equivariant $\mathcal{O}_Y$-module associated with the dual of the p-th symmetric tensor product $S^p(\mathfrak{g}/(\mathfrak{h}+\mathfrak{k}))^*$ with the coadjoint action of $H \cap K$. Let $\mathcal{T}_X$ be the sheaf of vector fields in X and let $\mathcal{T}_{X/Y}$ be the sheaf of vector fields in X tangent to Y, namely

$$\mathcal{T}_{X/Y} := \{\xi \in \mathcal{T}_X : \xi(\mathcal{I}_Y) \subset \mathcal{I}_Y\}.$$

Then $\xi \in \mathcal{T}_X$ operates on $\mathcal{O}_X$ and induces an $\mathcal{O}_Y$-homomorphism

$$\xi : i^{-1}(\mathcal{I}_Y/\mathcal{I}_Y^2) \to i^{-1}(\mathcal{O}_X/\mathcal{I}_Y) \simeq \mathcal{O}_Y.$$

This gives an isomorphism of locally free $\mathcal{O}_Y$-modules

$$i^{-1}(\mathcal{T}_X/\mathcal{T}_{X/Y}) \simeq \mathcal{H}om_{\mathcal{O}_Y}(i^{-1}(\mathcal{I}_Y/\mathcal{I}_Y^2), \mathcal{O}_Y),$$

which correspond to the normal bundle of Y in X.

We denote by $\mathcal{D}_X$ the ring of differential operators on X. Then $\mathcal{D}_X$ has the filtration given by

$$F_p\mathcal{D}_X := \{\xi \in \mathcal{D}_X : \xi(\mathcal{I}_Y^{p+1}) \subset \mathcal{I}_Y\},$$

which is called the filtration by normal degree with respect to i. A section of $F_p\mathcal{D}_X$ is locally written as $\sum \eta_1 \cdots \eta_r \xi_1 \dots \xi_q$, where $q \leq p$, $\xi_1, \dots, \xi_q \in \mathcal{T}_X$, and $\eta_1, \dots, \eta_r \in \mathcal{T}_{X/Y}$. Let $G_p\mathcal{D}_X (\subset \mathcal{D}_X)$ be the sheaf of differential operators on X with rank equal or less than p. For $D \in G_p\mathcal{D}_X$, the differential operator $D : \mathcal{O}_X \to \mathcal{O}_X$ induces an $\mathcal{O}_Y$-homomorphism

$$i^{-1}(\mathcal{I}_Y^p/\mathcal{I}_Y^{p+1}) \to i^{-1}(\mathcal{O}_X/\mathcal{I}_Y) \simeq \mathcal{O}_Y,$$

which we denote by $\gamma(D)$. Write

$$i^{-1}(\mathcal{I}_Y^p/\mathcal{I}_Y^{p+1})^\vee := \mathcal{H}om_{\mathcal{O}_Y}(i^{-1}(\mathcal{I}_Y^p/\mathcal{I}_Y^{p+1}), \mathcal{O}_Y)$$

for the dual of $i^{-1}(\mathcal{I}_Y^p/\mathcal{I}_Y^{p+1})$. The map $D \mapsto \gamma(D)$ gives an isomorphism of $\mathcal{O}_Y$-modules

$$i^{-1}G_p\mathcal{D}_X/i^{-1}(G_p\mathcal{D}_X \cap F_{p-1}\mathcal{D}_X) \simeq i^{-1}(\mathcal{I}_Y^p/\mathcal{I}_Y^{p+1})^\vee. \tag{3.3}$$

They are also isomorphic to the p-th symmetric tensor of the locally free $\mathcal{O}_Y$-module $i^{-1}(\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$.

Let $\mathcal{M}$ be a left $\mathcal{D}_Y$-module. The Lie algebra $\mathfrak{k}$ acts on $\mathcal{M}$ by η_Y for $\eta \in \mathfrak{k}$. Write Ω_X and Ω_Y for the canonical sheaves of X and Y, respectively. The push-forward by i is defined by

$$i_+\mathcal{M} := i_*((\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^\vee.$$

Here, we write i_* for the push-forward of $\mathcal{O}$-modules or $\mathbb{C}$-modules and i_+ for the push-forward of $\mathcal{D}$-modules. i^* denotes the pull-back of $\mathcal{O}$-modules. It follows from the definition that

$$i^{-1}i_+\mathcal{M} \simeq (\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X) \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee.$$

By using the filtration by normal degree, we define the $(i^{-1}\mathcal{O}_X)$-module

$$F_p i^{-1}i_+\mathcal{M} := (\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}F_p\mathcal{D}_X) \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee$$

for $p \geq 0$. This is well-defined because $\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}F_p\mathcal{D}_X$ is stable under the left $\mathcal{D}_Y$-action. We see that $i^{-1}F_p\mathcal{D}_X$ is a flat $(i^{-1}\mathcal{O}_X)$-module, $\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}F_p\mathcal{D}_X$ is a left flat $\mathcal{D}_Y$-module, and $i^{-1}\Omega_X^\vee$ is a flat $(i^{-1}\mathcal{O}_X)$-module. Hence the $(i^{-1}\mathcal{O}_X)$-modules $F_p i^{-1}i_+\mathcal{M}$ form a filtration of $i^{-1}i_+\mathcal{M}$.

Consider the restriction of the $\mathfrak{g}$-action on $i_+\mathcal{M}$ to $\mathfrak{k}$. For $\eta \in \mathfrak{k}$, the vector field η_X is tangent to Y. Hence the $\mathfrak{k}$-action stabilizes each $F_p i^{-1} i_+\mathcal{M}$ and it induces an action on the quotient $F_p i^{-1} i_+\mathcal{M}/F_{p-1} i^{-1} i_+\mathcal{M}$. Moreover, $F_p\mathcal{D}_X \cdot \mathcal{I}_Y \subset F_{p-1}\mathcal{D}_X$ implies that $i^{-1}\mathcal{I}_Y \cdot F_p i^{-1} i_+\mathcal{M} \subset F_{p-1} i^{-1} i_+\mathcal{M}$. Therefore $F_p i^{-1} i_+\mathcal{M}/F_{p-1} i^{-1} i_+\mathcal{M}$ carries an $\mathcal{O}_Y$-module structure. Write $\Omega_{X/Y} := \Omega_Y^\vee \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X$ for the relative canonical sheaf. The K-equivariant structures on the $\mathcal{O}_Y$-modules $\Omega_{X/Y}^\vee$ and $i^{-1}(\mathcal{I}^p/\mathcal{I}^{p+1})$ give $\mathfrak{k}$-actions on them.

Lemma 3.3. *There is an isomorphism of $\mathcal{O}_Y$-modules*

$$F_p i^{-1} i_+\mathcal{M}/F_{p-1} i^{-1} i_+\mathcal{M} \simeq \mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_{X/Y}^\vee \otimes_{\mathcal{O}_Y} i^{-1}(\mathcal{I}_Y^p/\mathcal{I}_Y^{p+1})^\vee$$

that commutes with the actions of $\mathfrak{k}$. Here, the $\mathfrak{k}$-action on the right side is given by the tensor product of the action on each factors defined above.

Proof. The inverse image $i^*\mathcal{D}_X := \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X$ of $\mathcal{D}_X$ in the category of $\mathcal{O}$-modules has a left $\mathcal{D}_Y$-module structure. The action map

$$\mathcal{D}_Y \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X) \to \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X$$

induces a morphism of left $\mathcal{D}_Y$-modules

$$\begin{aligned} &\mathcal{D}_Y \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}(G_p\mathcal{D}_X/(G_p\mathcal{D}_X \cap F_{p-1}\mathcal{D}_X))) \\ &\to \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}(F_p\mathcal{D}_X/F_{p-1}\mathcal{D}_X). \end{aligned} \tag{3.4}$$

We give the inverse map of (3.4). Any section of $F_p\mathcal{D}_X/F_{p-1}\mathcal{D}_X$ is represented by a sum of section of the form $\eta_1 \cdots \eta_r \xi_1 \cdots \xi_p$ for $\xi_1, \dots, \xi_p \in \mathcal{T}_X$ and $\eta_1, \dots, \eta_r \in \mathcal{T}_{X/Y}$. The inverse map

$$\begin{aligned} &\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}(F_p\mathcal{D}_X/F_{p-1}\mathcal{D}_X) \\ &\to \mathcal{D}_Y \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}(G_p\mathcal{D}_X/(G_p\mathcal{D}_X \cap F_{p-1}\mathcal{D}_X))) \end{aligned}$$

is given by

$$f \otimes \eta_1 \cdots \eta_r \xi_1 \cdots \xi_p \mapsto f(\eta_1)|_Y \cdots (\eta_r)|_Y \otimes (1 \otimes \xi_1 \cdots \xi_p).$$

Hence (3.4) is an isomorphism.

By using (3.3) and (3.4), we obtain isomorphisms of $\mathcal{O}_Y$-modules:

$$\begin{aligned} &F_p i^{-1} i_+\mathcal{M}/F_{p-1} i^{-1} i_+\mathcal{M} \\ &\simeq (\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}(F_p\mathcal{D}_X/F_{p-1}\mathcal{D}_X)) \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee \\ &\simeq (\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} (\mathcal{D}_Y \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}(G_p\mathcal{D}_X/(G_p\mathcal{D}_X \cap F_{p-1}\mathcal{D}_X)))) \\ &\qquad\qquad \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee \\ &\simeq (\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{O}_Y} i^{-1}(G_p\mathcal{D}_X/(G_p\mathcal{D}_X \cap F_{p-1}\mathcal{D}_X)) \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee \\ &\simeq \mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_{X/Y}^\vee \otimes_{\mathcal{O}_Y} i^{-1}(\mathcal{I}_Y^p/\mathcal{I}_Y^{p+1})^\vee. \end{aligned} \tag{3.5}$$

We now show that this map commutes with the $\mathfrak{k}$-actions. Take a section

$$(m \otimes \omega) \otimes (1 \otimes D) \otimes \omega' \in (\mathcal{M} \otimes_{\mathcal{O}} \Omega_Y) \otimes_{\mathcal{D}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}} i^{-1}F_p\mathcal{D}_X) \otimes_{i^{-1}\mathcal{O}} i^{-1}\Omega_X^\vee$$

for $m \in \mathcal{M}$, $\omega \in \Omega_Y$, $D \in G_p\mathcal{D}_X$, and $\omega' \in \Omega_X^\vee$. Since any section of $F_p i^{-1} i_+\mathcal{M}/F_{p-1} i^{-1} i_+\mathcal{M}$ is represented by a sum of sections of this form, it is enough to see the commutativity for this section. Under the isomorphisms (3.5), the section $(m\otimes\omega)\otimes(1\otimes D)\otimes\omega'$

corresponds to $m \otimes (\omega \otimes \omega') \otimes \gamma(D) \in \mathcal{M} \otimes_{\mathcal{O}} \Omega_{X/Y}^{\vee} \otimes_{\mathcal{O}} i^{-1}(\mathcal{I}^p/\mathcal{I}^{p+1})^{\vee}$. For $\eta \in \mathfrak{k}$, the $\mathfrak{k}$-action on $i^{-1}i_+\mathcal{M}$ is given by

$$\begin{aligned}
&(m \otimes \omega) \otimes (1 \otimes D) \otimes \omega' \\
\mapsto\ & (m \otimes \omega) \otimes (1 \otimes D(-\eta_X)) \otimes \omega' + (m \otimes \omega) \otimes (1 \otimes D) \otimes \eta_X \omega' \\
=\ & (m \otimes \omega) \otimes (1 \otimes (-\eta_X) D) \otimes \omega' + (m \otimes \omega) \otimes (1 \otimes [\eta_X, D]) \otimes \omega' \\
& \qquad + (m \otimes \omega) \otimes (1 \otimes D) \otimes \eta_X \omega'.
\end{aligned}$$

Since $\eta_X|_Y = \eta_Y$, it follows that

$$\begin{aligned}
&(m \otimes \omega) \otimes (1 \otimes (-\eta_X) D) \otimes \omega' = (m \otimes \omega)(-\eta_Y) \otimes (1 \otimes D) \otimes \omega' \\
&\quad = (\eta_Y m \otimes \omega) \otimes (1 \otimes D) \otimes \omega' + (m \otimes \omega(-\eta_Y)) \otimes (1 \otimes D) \otimes \omega'.
\end{aligned}$$

As a result, the action of η is given by

$$\begin{aligned}
&\eta \cdot ((m \otimes \omega) \otimes (1 \otimes D) \otimes \omega') \\
&= (\eta_Y m \otimes \omega) \otimes (1 \otimes D) \otimes \omega' + (m \otimes \omega(-\eta_Y)) \otimes (1 \otimes D) \otimes \omega' \\
&+ (m \otimes \omega) \otimes (1 \otimes [\eta_X, D]) \otimes \omega' + (m \otimes \omega) \otimes (1 \otimes D) \otimes \eta_X \omega'.
\end{aligned}$$

Since $[\eta_X, D] \in G_p\mathcal{D}_X$, the section $\eta \cdot ((m \otimes \omega) \otimes (1 \otimes D) \otimes \omega')$ corresponds to

$$\begin{aligned}
&\eta_Y m \otimes (\omega \otimes \omega') \otimes \gamma(D) + m \otimes \eta_Y(\omega \otimes \omega') \otimes \gamma(D) \\
&\qquad + m \otimes (\omega \otimes \omega') \otimes \gamma([\eta_X, D]).
\end{aligned}$$

Thus, the commutativity follows from $\gamma([\eta_X, D]) = \eta \cdot \gamma(D)$. □

In the rest of this section, we assume that K and $H \cap K$ are complex reductive linear algebraic groups. In particular, $Y := K/(H \cap K)$ is an affine variety by [18, §I.2].

We assume moreover that there exists a K-equivariant isomorphism of $\mathcal{O}_Y$-modules: $\Omega_Y \simeq \mathcal{O}_Y$, or equivalently, the $(H \cap K)$-module $\bigwedge^{\mathrm{top}}(\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{k}))$ with the adjoint action is trivial. This assumption automatically holds if $H \cap K$ is connected.

Let V be an H-module. Then V is written as a union of finite-dimensional H-submodules and has a structure of $(\mathfrak{h}, H \cap K)$-module. Define the $(\mathfrak{g}, K)$-module $R(\mathfrak{g}, K) \otimes_{R(\mathfrak{h}, H \cap K)} V$ as in Section 2.

Let $\mathcal{V}_X$ be the $\mathcal{O}_X$-module associated with the H-module V. Then the G-equivariant structures of $\mathcal{V}_X$ and Ω_X induce $(\mathfrak{g}, K)$-actions on them.

The next lemma relates these two modules.

Lemma 3.4. *Under the assumptions above, there is an isomorphism of $(\mathfrak{g}, K)$-modules*

$$R(\mathfrak{g}, K) \otimes_{R(\mathfrak{h}, H \cap K)} V \xrightarrow{\sim} \Gamma(X, i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega_X \otimes_{\mathcal{O}_X} \mathcal{V}_X),$$

where the actions of $\mathfrak{g}$ and K on the right side are given by the tensor product of three factors.

Proof. With the identification $\Omega_Y \simeq \mathcal{O}_Y$, we have

$$i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega_X \simeq i_*(\mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X).$$

Hence

$$i^{-1}(i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega_X \otimes_{\mathcal{O}_X} \mathcal{V}_X) \simeq \mathcal{O}_Y \otimes_{\mathcal{D}_Y} (i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X).$$

Using the right $(i^{-1}\mathcal{D}_X)$-module structure of $i^*\mathcal{D}_X$, we define a $\mathfrak{g}$-action ρ on the sheaf $i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X$ by

$$\rho(\xi)(D \otimes v) := D(-\xi_X) \otimes v + D \otimes \xi v$$

for $\xi \in \mathfrak{g}$, $D \in i^*\mathcal{D}_X$, and $v \in \mathcal{V}_X$. Moreover, the sheaf $i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X$ is K-equivariant. We denote this K-action and also its infinitesimal $\mathfrak{k}$-action by ν. Using the $(\mathcal{D}_Y, i^{-1}\mathcal{D}_X)$-bimodule structure on $i^*\mathcal{D}_X$, the $\mathfrak{k}$-action ν is given by

$$\nu(\eta)(D \otimes v) = \eta_Y D \otimes v - D\eta_X \otimes v + D \otimes \eta v$$

for $\eta \in \mathfrak{k}$. Then $\Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X)$ is a weak Harish-Chandra module in the sense of [17], namely,

$$\nu(k)\rho(\xi)\nu(k^{-1}) = \rho(\mathrm{Ad}(k)\xi) \tag{3.6}$$

for $k \in K$ and $\xi \in \mathfrak{g}$. Put $\omega(\eta) := \nu(\eta) - \rho(\eta)$ for $\eta \in \mathfrak{k}$. Then $\omega(\eta)$ is given by

$$\omega(\eta)(D \otimes v) = \eta_Y D \otimes v.$$

Since Y is an affine variety, $\Gamma(Y, \mathcal{D}_Y)$ is generated by $U(\mathfrak{k})$ as an $\mathcal{O}(Y)$-algebra. Therefore,

$$\begin{aligned}
&\Gamma(X, i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega_X \otimes_{\mathcal{O}_X} \mathcal{V}_X) \\
&\simeq \mathcal{O}(Y) \otimes_{\Gamma(Y,\mathcal{D}_Y)} \Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X) \\
&\simeq \Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X) \,/\, \omega(\mathfrak{k})\Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X).
\end{aligned}$$

Let $e \in K$ be the identity element. Write $o := e(H \cap K) \in Y$ for the base point and $i_{o,Y} : \{o\} \to Y$ for the immersion. Let $\mathcal{I}_o$ be the maximal ideal of $\mathcal{O}_Y$ corresponding to o. The geometric fiber of $i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X$ at o is given by

$$\begin{aligned}
W &:= (i_{o,Y})^*(i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X) \\
&\simeq \Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X) \,/\, \mathcal{I}_o(Y)\Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X).
\end{aligned}$$

The actions ρ and ν on $i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X$ induce a $\mathfrak{g}$-action ρ_o and an $(H \cap K)$-action ν_o on W. With these actions, W becomes a $(\mathfrak{g}, H \cap K)$-module. To show this, it is enough to see that ρ_o and ν_o agree on $\mathfrak{h} \cap \mathfrak{k}$. This follows from

$$\omega(\eta)\Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X) \subset \mathcal{I}_o(Y)\Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X)$$

for $\eta \in \mathfrak{h} \cap \mathfrak{k}$.

We claim that $W \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V$ as a $(\mathfrak{g}, H \cap K)$-module. Put $i_{o,X} := i \circ i_{o,Y}$. Then

$$\begin{aligned}
W &\simeq (i_{o,X})^*\mathcal{D}_X \otimes_{(i_{o,X})^{-1}\mathcal{O}_X} (i_{o,X})^{-1}\mathcal{V}_X \\
&\simeq (i_{o,X})^{-1}((i_{o,X})_+\mathcal{O}_{\{o\}} \otimes_{\mathcal{O}_X} \Omega_X) \otimes_{(i_{o,X})^{-1}\mathcal{O}_X} (i_{o,X})^{-1}\mathcal{V}_X.
\end{aligned}$$

Let $\{F_p\mathcal{D}_X\}$ be the filtration by normal degree with respect to $i_{o,X}$. Define the filtration

$$F_pW := (i_{o,X})^* F_p\mathcal{D}_X \otimes_{(i_{o,X})^{-1}\mathcal{O}_X} (i_{o,X})^{-1}\mathcal{V}_X$$

of W. Then F_pW is $(\mathfrak{h}, H \cap K)$-stable and there is an isomorphism of $(\mathfrak{h}, H \cap K)$-modules

$$F_pW/F_{p-1}W \simeq (i_{o,X})^{-1}(\mathcal{I}_o^p/\mathcal{I}_o^{p+1})^\vee \otimes V$$

by Lemma 3.3. The isomorphism $F_0W \simeq V$ induces a $(\mathfrak{g}, H \cap K)$-homomorphism $\varphi : U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V \to W$. Let $U_p(\mathfrak{g})$ be the standard filtration of $U(\mathfrak{g})$. Then

$(U_p(\mathfrak{g})U(\mathfrak{h}))\otimes_{U(\mathfrak{h})} V$ is a filtration of the $(\mathfrak{h}, H\cap K)$-module $U(\mathfrak{g})\otimes_{U(\mathfrak{h})} V$ and there is an isomorphism of $(\mathfrak{h}, H\cap K)$-modules:

$$(U_p(\mathfrak{g})U(\mathfrak{h}))\otimes_{U(\mathfrak{h})} V \,/\, (U_{p-1}(\mathfrak{g})U(\mathfrak{h}))\otimes_{U(\mathfrak{h})} V \simeq S^p(\mathfrak{g}/\mathfrak{h})\otimes V.$$

In view of the proof of Lemma 3.3, we see that the map on the successive quotient

$$\varphi_p : (U_p(\mathfrak{g})U(\mathfrak{h}))\otimes_{U(\mathfrak{h})} V \,/\, (U_{p-1}(\mathfrak{g})U(\mathfrak{h}))\otimes_{U(\mathfrak{h})} V \to F_pW/F_{p-1}W$$

induced by φ is an isomorphism. Hence φ is an isomorphism.

As a K-equivariant $\mathcal{O}_Y$-module, $i^*\mathcal{D}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X$ is isomorphic to the $\mathcal{O}_Y$-module $\mathcal{W}_Y$ associated with the $(H\cap K)$-module W. Hence we can see global sections $\Gamma(Y, i^*\mathcal{D}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X)$ as W-valued regular functions on K. Let f be a W-valued regular function on K such that $f(kh) = \nu_o(h^{-1})f(k)$ for $k\in K$ and $h \in H\cap K$. The $\mathfrak{g}$-action ρ at e is given by $(\rho(\xi)f)(e) = \rho_o(\xi)(f(e))$. Hence (3.6) implies that

$$(\rho(\xi)f)(k) = (\nu(k)\rho(\mathrm{Ad}(k^{-1})\xi)\nu(k^{-1})f)(k) = \rho_o(\mathrm{Ad}(k^{-1})\xi)(f(k)).$$

Let $\xi_1,\dots,\xi_n$ be a basis of $\mathfrak{g}$ and write $\xi^1,\dots,\xi^n\in\mathfrak{g}^*$ for its dual basis. Under the isomorphism $\Gamma(Y,\mathcal{W}_Y)\simeq R(K)\otimes_{R(H\cap K)} W$ given in Lemma 3.1, the $\mathfrak{g}$-action ρ on $R(K)\otimes_{R(H\cap K)} W$ is given by

$$\rho(\xi)(S\otimes w) = \sum_{i=1}^{n} \langle \xi^i, \mathrm{Ad}(\cdot)^{-1}\xi\rangle S\otimes \rho_o(\xi_i)w \tag{3.7}$$

for $S\in R(K)$ and $w\in W$. If we define ρ on $R(K)\otimes_{\mathbb{C}} W$ by this equation, then ρ commutes with the canonical surjective map

$$p : R(K)\otimes_{\mathbb{C}} W \to R(K)\otimes_{R(H\cap K)} W.$$

The K-action ν is given by the left translation of $R(K)$:

$$\nu(k)(S\otimes w) = (kS)\otimes w.$$

Hence ν also lifts to the action on $R(K)\otimes_{\mathbb{C}} W$ and commutes with p. Let $\eta_1,\cdots,\eta_m$ be a basis of $\mathfrak{k}$ and write $\eta^1,\cdots,\eta^m\in\mathfrak{k}^*$ for its dual basis. Define the regular functions α^i_j and β^j_i on K with respect to η_i as in (3.1). Then the $\mathfrak{k}$-action ω is given by

$$\begin{aligned}\omega(\eta_j)(S\otimes w) &= \nu(\eta_j)(S\otimes w) - \rho(\eta_j)(S\otimes w)\\ &= ((\eta_j)^L_K S)\otimes w - \sum_{i=1}^{m}\alpha^i_j S\otimes \rho_o(\eta_i)w.\end{aligned}$$

Here, we identify $R(K)$ with $\mathcal{O}(K)$, and give actions of differential operators on K.

We have

$$\begin{aligned}&\Gamma(X, i_+\mathcal{O}_Y\otimes_{\mathcal{O}_X}\Omega_X\otimes_{\mathcal{O}_X}\mathcal{V}_X)\\ \simeq\ &\Gamma(Y, i^*\mathcal{D}_X\otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X)/\omega(\mathfrak{k})\Gamma(Y, i^*\mathcal{D}_X\otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{V}_X)\\ \simeq\ &(R(K)\otimes_{R(H\cap K)} W)/\omega(\mathfrak{k})(R(K)\otimes_{R(H\cap K)} W).\end{aligned}$$

We note that the $\mathfrak{k}$-actions ρ and ν agree on the quotient $(R(K)\otimes_{R(H\cap K)} W)/\omega(\mathfrak{k})(R(K)\otimes_{R(H\cap K)} W)$ and hence it becomes a $(\mathfrak{g}, K)$-module.

The equation $\sum_{j=1}^m \alpha^i_j\beta^j_k = \delta^i_k$ implies that $\omega(\mathfrak{k})(R(K)\otimes_{\mathbb{C}} W)$ is generated by the elements of the form $\sum_{j=1}^m \omega(\eta_j)(\beta^j_k S\otimes w)$ for $S\in R(K)$ and $w\in W$. We observe

from (3.2) that $\sum_{j=1}^{m}(\eta_j)_K^L(\beta_k^j) = 0$ because $\mathrm{Trace}\,\mathrm{ad}(\cdot) = 0$ for the reductive Lie algebra $\mathfrak{k}$. Therefore,

$$(\eta_k)_K^R = -\sum_{j=1}^{m}\beta_k^j(\eta_j)_K^L = -\sum_{j=1}^{m}(\eta_j)_K^L\beta_k^j$$

as differential operators on K. Then

$$\begin{aligned}\sum_{j=1}^{m}\omega(\eta_j)(\beta_k^j S\otimes w) &= \sum_{j=1}^{m}(\eta_j)_K^L\beta_k^j S\otimes w + \sum_{i,j=1}^{m}(\alpha_j^i\beta_k^j S\otimes\rho_o(\eta_i)w)\\ &= -(\eta_k)_K^R S\otimes w + S\otimes\rho_o(\eta_k)w.\end{aligned}$$

Consequently, the kernel of the map

$$R(K)\otimes_{\mathbb{C}} W \to (R(K)\otimes_{R(H\cap K)} W)/\omega(\mathfrak{k})(R(K)\otimes_{R(H\cap K)} W)$$

is generated by $\mathrm{Ker}\, p$ and $-(\eta_k)_K^R S\otimes w + S\otimes\rho_o(\eta_k)w$. Hence

$$\begin{aligned}&(R(K)\otimes_{R(H\cap K)} W)/\omega(\mathfrak{k})(R(K)\otimes_{R(H\cap K)} W)\\ &\simeq R(K)\otimes_{R(\mathfrak{k},H\cap K)} W\\ &\simeq R(\mathfrak{g},K)\otimes_{R(\mathfrak{g},H\cap K)} W.\end{aligned}$$

From (3.7), we see that the isomorphism

$$(R(K)\otimes_{R(H\cap K)} W)/\omega(\mathfrak{k})(R(K)\otimes_{R(H\cap K)} W)\simeq R(\mathfrak{g},K)\otimes_{R(\mathfrak{g},H\cap K)} W$$

commutes with the $(\mathfrak{g},K)$-actions. Therefore,

$$\begin{aligned}\Gamma(X, i_+\mathcal{O}_Y\otimes_{\mathcal{O}_X}\Omega_X\otimes_{\mathcal{O}_X}\mathcal{V}_X) &\simeq R(\mathfrak{g},K)\otimes_{R(\mathfrak{g},H\cap K)} W\\ &\simeq R(\mathfrak{g},K)\otimes_{R(\mathfrak{h},H\cap K)} V\end{aligned}$$

and the lemma is proved. □

4. Localization of the Cohomological Induction

In this section, we construct cohomologically induced modules on flag varieties. Let G_0 be a connected real linear reductive Lie group with Lie algebra $\mathfrak{g}_0$ and $\mathfrak{q}$ a θ-stable parabolic subalgebra as in Section 2. We define the complexification G of G_0 as a complex reductive linear algebraic group. Write $\overline{Q}$ for the parabolic subgroup of G with Lie algebra $\bar{\mathfrak{q}}$.

Suppose that V is a $\overline{Q}$-module and use the same letter V for the underlying $(\bar{\mathfrak{q}}, L\cap K)$-module. In Section 2, we define the cohomologically induced module

$$(\Pi_{L\cap K}^K)_s(U(\mathfrak{g})\otimes_{U(\bar{\mathfrak{q}})}(V\otimes\mathbb{C}_{2\rho(\mathfrak{u})})),$$

where $s = \dim(\mathfrak{u}\cap\mathfrak{k})$.

Let $X := G/\overline{Q}$ and $Y := K/(\overline{Q}\cap K)$, which are the partial flag varieties of G and K, respectively. The inclusion map $i : Y \to X$ is a closed immersion. Let $i_+\mathcal{O}_Y$ be the push-forward of $\mathcal{O}_Y$ in the category of $\mathcal{D}$-modules. We write $\mathcal{V}_X$ for the G-equivariant $\mathcal{O}_X$-module associated with the $\overline{Q}$-module V as in Section 3.

The next theorem relates the cohomologically induced module and the $\mathcal{O}_X$-module $i_+\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{V}_X$. This theorem is similar to that in [5], but differs in the following three ways. First, we assume that $\mathfrak{q}$ is a θ-stable parabolic subalgebra and hence Y is a closed subvariety of the partial flag variety X, while in [5], X is a complete flag variety and Y is an arbitrary K-orbit. Second, we assume that V is a $\overline{Q}$-module and consider the $\mathcal{O}_X$-module $i_+\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{V}_X$ with $(\mathfrak{g},K)$-action. On the

other hand, V is a one-dimensional $(\mathfrak{l}, L\cap K)$-module and the corresponding twisted $\mathcal{D}$-module was used in [5]. Third, we adopt the functor $P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K}$ for cohomologically induced modules instead of $I^{\mathfrak{g},K}_{\mathfrak{q},L\cap K}$. As a result, the dual in the isomorphism in [5] does not appear in Theorem 4.1.

Theorem 4.1. *Let V be a $\overline{Q}$-module. Then there is an isomorphism*

$$(\Pi^K_{L\cap K})_{s-i}(U(\mathfrak{g})\otimes_{U(\bar{\mathfrak{q}})}(V\otimes\mathbb{C}_{2\rho(\mathfrak{u})}))\simeq \mathrm{H}^i(X, i_+\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{V}_X)$$

of $(\mathfrak{g},K)$-modules.

Proof. Let $\widetilde{X} := G/L$ and $\widetilde{Y} := K/(L\cap K)$. We have the commutative diagram:

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\tilde{i}"] \arrow[d] & \widetilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

where the maps are defined canonically. Denote by $\mathcal{T}_{\widetilde{X}/X}$ the sheaf of local vector fields on $\widetilde{X}$ tangent to the fiber of π and denote by $\Omega_{\widetilde{X}/X}$ the top exterior product of its dual $\mathcal{T}^\vee_{\widetilde{X}/X}$. Then $\Omega_{\widetilde{X}/X}$ is canonically isomorphic to $\Omega_{\widetilde{X}}\otimes_{\mathcal{O}_{\widetilde{X}}}\pi^*(\Omega^\vee_X)$. Consider the complex of $(\pi^{-1}\mathcal{D}_X)$-modules

$$\mathcal{C}^{-d} := \tilde{i}_+\mathcal{O}_{\widetilde{Y}}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\bigwedge^d\mathcal{T}_{\widetilde{X}/X}.$$

The boundary map $\mathcal{C}^{-d}\to\mathcal{C}^{-d+1}$ is given by

$$\begin{aligned} &f\otimes\omega\otimes\xi_1\wedge\cdots\wedge\xi_d \\ \mapsto &\sum_i(-1)^{i+1}\big(-\xi_i f\otimes\omega\otimes\xi_1\wedge\cdots\wedge\widehat{\xi_i}\wedge\cdots\wedge\xi_d \\ &\qquad\qquad\qquad\qquad + f\otimes\omega\xi_i\otimes\xi_1\wedge\cdots\wedge\widehat{\xi_i}\wedge\cdots\wedge\xi_d\big) \\ +&\sum_{i<j}(-1)^{i+j}\big(f\otimes\omega\otimes[\xi_i,\xi_j]\wedge\xi_1\wedge\cdots\wedge\widehat{\xi_i}\wedge\cdots\wedge\widehat{\xi_j}\wedge\cdots\wedge\xi_d\big), \end{aligned}$$

where $f\in\tilde{i}_+\mathcal{O}_{\widetilde{Y}}$, $\omega\in\Omega_{\widetilde{X}/X}$ and $\xi_1,\dots,\xi_d\in\mathcal{T}_{\widetilde{X}/X}$. Since $\Omega_{\widetilde{X}/X}$ and $\mathcal{T}_{\widetilde{X}/X}$ are G-equivariant, $\mathfrak{g}$ acts on them by differential. The action of $\mathfrak{g}$ on $\mathcal{C}^d$ is given by the tensor product of the actions on $\tilde{i}_+\mathcal{O}_{\widetilde{Y}}$, $\Omega_{\widetilde{X}/X}$ and $\mathcal{T}_{\widetilde{X}/X}$.

By an argument in [5], we have a quasi-isomorphism of the complexes of $\mathcal{D}_X$-modules $\pi_*\mathcal{C}^\bullet\simeq(i_+\mathcal{O}_Y)[s]$, where $[s]$ denotes the shift by s. Then the projection formula gives a quasi-isomorphism of complexes of $\mathcal{O}_X$-modules

$$\pi_*(\mathcal{C}^\bullet\otimes_{\pi^{-1}\mathcal{O}_X}\pi^{-1}\mathcal{V}_X)\simeq i_+\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{V}_X[s].$$

The isomorphism $\Omega_{\widetilde{X}/X}\simeq\Omega_{\widetilde{X}}\otimes_{\mathcal{O}_{\widetilde{X}}}\pi^*(\Omega^\vee_X)$ gives

$$\mathcal{C}^{-d}\otimes_{\pi^{-1}\mathcal{O}_X}\pi^{-1}\mathcal{V}_X\simeq\tilde{i}_+\mathcal{O}_{\widetilde{Y}}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}}\otimes_{\mathcal{O}_{\widetilde{X}}}\bigwedge^d\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\pi^*(\mathcal{V}_X\otimes_{\mathcal{O}_X}\Omega^\vee_X).$$

The boundary map ∂ on the right side is given by

$$\begin{aligned} & f \otimes \xi_1 \wedge \cdots \wedge \xi_d \otimes v \\ \mapsto\ & \sum_i (-1)^{i+1} \big(f\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_d \otimes v \\ & \qquad\qquad\qquad\qquad - f \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_d \otimes \xi_i v \big) \\ +\ & \sum_{i<j} (-1)^{i+j} \big(f \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_d \otimes v \big) \end{aligned}$$

for $f \in \tilde{\imath}_+ \mathcal{O}_{\widetilde{Y}} \otimes \Omega_{\widetilde{X}}$, $\xi_1, \ldots, \xi_d \in \mathcal{T}_{\widetilde{X}/X}$, and $v \in \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^\vee)$. Here, the action of $\xi_i \in \mathcal{T}_{\widetilde{X}/X}$ on $\tilde{\imath}_+ \mathcal{O}_{\widetilde{Y}} \otimes \Omega_{\widetilde{X}}$ is defined by the right $\mathcal{D}_{\widetilde{X}}$-module structure of $\tilde{\imath}_+ \mathcal{O}_{\widetilde{Y}} \otimes \Omega_{\widetilde{X}}$, and the action of ξ_i on

$$\pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^\vee) := \mathcal{O}_{\widetilde{X}} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^\vee)$$

is given by the action on the first factor $\mathcal{O}_{\widetilde{X}}$ of the right side. Since $\widetilde{X}$ is affine, we have an isomorphism of $(\mathfrak{g}, K)$-modules

$$\begin{aligned} & \mathrm{H}^{i-s}\Big(\Gamma\big(\widetilde{X},\, \tilde{\imath}_+ \mathcal{O}_{\widetilde{Y}} \otimes_{\mathcal{O}_{\widetilde{X}}} \Omega_{\widetilde{X}} \otimes_{\mathcal{O}_{\widetilde{X}}} \bigwedge^{\bullet} \mathcal{T}_{\widetilde{X}/X} \otimes_{\mathcal{O}_{\widetilde{X}}} \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^\vee) \big) \Big) \\ \simeq\ & \mathrm{H}^i(X,\, i_+ \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{V}_X). \end{aligned}$$

We now compute the cohomologically induced module $(\Pi^K_{L\cap K})_{s-i}(U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} (V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}))$. The standard complex of $\bar{\mathfrak{u}}$ is the complex $U(\bar{\mathfrak{u}}) \otimes \bigwedge^\bullet \bar{\mathfrak{u}}$ with the boundary map

$$\begin{aligned} D \otimes \xi_1 \wedge \cdots \wedge \xi_d \mapsto & \sum_{i=1}^{d} (-1)^{i+1} \big(D\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_d \big) \\ & + \sum_{i<j} (-1)^{i+j} \big(D \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_d \big) \end{aligned}$$

for $D \in U(\bar{\mathfrak{u}})$ and $\xi_1, \ldots, \xi_d \in \bar{\mathfrak{u}}$. This gives a left resolution of the trivial $\bar{\mathfrak{u}}$-module:

$$U(\bar{\mathfrak{u}}) \otimes \bigwedge^{\bullet} \bar{\mathfrak{u}} \to \mathbb{C}.$$

Since $U(\bar{\mathfrak{u}}) \simeq U(\bar{\mathfrak{q}})/U(\bar{\mathfrak{q}})\mathfrak{l}$, we have an isomorphism

$$U(\bar{\mathfrak{q}}) \otimes_{U(\mathfrak{l})} \bigwedge^{d} \bar{\mathfrak{u}} \simeq U(\bar{\mathfrak{u}}) \otimes_{\mathbb{C}} \bigwedge^{d} \bar{\mathfrak{u}}.$$

Hence we have a left resolution of the trivial $(\bar{\mathfrak{q}}, L\cap K)$-modules:

$$U(\bar{\mathfrak{q}}) \otimes_{U(\mathfrak{l})} \bigwedge^{d} \bar{\mathfrak{u}} \to \mathbb{C}.$$

By taking tensor product with $V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}$, we get a resolution of the $(\bar{\mathfrak{q}}, L\cap K)$-module $V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}$:

$$U(\bar{\mathfrak{q}}) \otimes_{U(\mathfrak{l})} (\bigwedge^{\bullet} \bar{\mathfrak{u}} \otimes V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}) \simeq (U(\bar{\mathfrak{q}}) \otimes_{U(\mathfrak{l})} \bigwedge^{\bullet} \bar{\mathfrak{u}}) \otimes (V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}) \to V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}.$$

Therefore, we have a resolution of the $(\mathfrak{g}, L\cap K)$-module $U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} (V \otimes \mathbb{C}_{2\rho(\mathfrak{u})})$:

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} (\bigwedge^{\bullet} \bar{\mathfrak{u}} \otimes V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}) \to U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} (V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}),$$

where the boundary map ∂' is given by

$$
\begin{aligned}
& D \otimes \xi_1 \wedge \cdots \wedge \xi_d \otimes v \\
\mapsto\ & \sum_{i=1}^{d} (-1)^{i+1} \big(D\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_d \otimes v \\
& \qquad\qquad\qquad\qquad - D \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_d \otimes \xi_i v \big) \\
+\ & \sum_{i<j} (-1)^{i+j} \big(D \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_d \otimes v \big)
\end{aligned}
$$

for $D \in U(\mathfrak{g})$, $\xi_1, \ldots, \xi_d \in \bar{\mathfrak{u}}$, and $v \in V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}$.

Lemma 4.2. *For any $(\mathfrak{l}, L \cap K)$-module W, the $(\mathfrak{g}, L \cap K)$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W$ is $\Pi^K_{L\cap K}$-acyclic.*

Proof. By [8, Proposition 2.115], $(P^{\mathfrak{g},K}_{\mathfrak{g},L\cap K})_j(U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W) \simeq (P^{\mathfrak{k},K}_{\mathfrak{k},L\cap K})_j(U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W)$ as a K-module. Hence it is enough to show that $(P^{\mathfrak{k},K}_{\mathfrak{k},L\cap K})_j(U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W) = 0$ for $j > 0$. Let $U_p(\mathfrak{g})$ be the standard filtration of $U(\mathfrak{g})$ and let $U'_p(\mathfrak{g}) := U(\mathfrak{k})U_p(\mathfrak{g})U(\mathfrak{l}) \subset U(\mathfrak{g})$ for $p \geq 0$. Then $U'_p(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W$ is a filtration of the $(\mathfrak{k}, L\cap K)$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W$ and it follows that

$$U'_p(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W \,/\, U'_{p-1}(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W \simeq U(\mathfrak{k}) \otimes_{U(\mathfrak{l}\cap\mathfrak{k})} (S^p(\mathfrak{g}/(\mathfrak{l}+\mathfrak{k})) \otimes W).$$

Since

$$\mathrm{Hom}_{\mathfrak{k},L\cap K}(U(\mathfrak{k}) \otimes_{U(\mathfrak{l}\cap\mathfrak{k})} (S^p(\mathfrak{g}/(\mathfrak{l}+\mathfrak{k})) \otimes W), \,\cdot\,) \simeq \mathrm{Hom}_{L\cap K}(S^p(\mathfrak{g}/(\mathfrak{l}+\mathfrak{k})) \otimes W, \,\cdot\,),$$

we see that $U'_p(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W \,/\, U'_{p-1}(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W$ is a projective $(\mathfrak{k}, L\cap K)$-module. Then we see inductively that $U'_p(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W$ is also a projective $(\mathfrak{k}, L \cap K)$-module and in particular $P^{\mathfrak{k},K}_{\mathfrak{k},L\cap K}$-acyclic. As a consequence,

$$
\begin{aligned}
(P^{\mathfrak{k},K}_{\mathfrak{k},L\cap K})_j(U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W) &= (P^{\mathfrak{k},K}_{\mathfrak{k},L\cap K})_j \varinjlim_p (U'_p(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W) \\
&= \varinjlim_p (P^{\mathfrak{k},K}_{\mathfrak{k},L\cap K})_j (U'_p(\mathfrak{g}) \otimes_{U(\mathfrak{l})} W) = 0
\end{aligned}
$$

for $j > 0$. $\square$

From the lemma, we conclude that

$$(\Pi^K_{L\cap K})_{s-i}(U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} (V \otimes \mathbb{C}_{2\rho(\mathfrak{u})})) \simeq \mathrm{H}^{i-s}(\Pi^K_{L\cap K}(U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} (\textstyle\bigwedge^{\bullet} \bar{\mathfrak{u}} \otimes V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}))).$$

To complete the proof of Theorem 4.1, it is enough to give an isomorphism of the complexes of $(\mathfrak{g}, K)$-modules:

$$
\begin{aligned}
\Gamma\big(\widetilde{X},\ \tilde{\imath}_+\mathcal{O}_{\widetilde{Y}} \otimes_{\mathcal{O}_{\widetilde{X}}} \Omega_{\widetilde{X}} \otimes_{\mathcal{O}_{\widetilde{X}}} \textstyle\bigwedge^{\bullet} \mathcal{T}_{\widetilde{X}/X} \otimes_{\mathcal{O}_{\widetilde{X}}} \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^\vee)\big) \\
\xrightarrow{\sim} R(\mathfrak{g}, K) \otimes_{R(\mathfrak{l}, L\cap K)} (\textstyle\bigwedge^{\bullet} \bar{\mathfrak{u}} \otimes V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}).
\end{aligned}
\tag{4.1}
$$

Let $o := e(L\cap K) \in \widetilde{Y}$ be the base point and $i_o : \{o\} \to \widetilde{Y}$ the immersion. Define the complex of left $\mathcal{D}_{\widetilde{Y}}$-modules

$$\tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \tilde{\imath}^{-1}\big(\textstyle\bigwedge^{\bullet} \mathcal{T}_{\widetilde{X}/X} \otimes_{\mathcal{O}_{\widetilde{X}}} \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^\vee)\big),$$

where the boundary map

$$\begin{aligned}\partial : \tilde{\imath}^{\,*}\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{\,-1}\mathcal{O}_{\widetilde{X}}} \tilde{\imath}^{\,-1}\big(\bigwedge^{d} \mathcal{T}_{\widetilde{X}/X} \otimes_{\mathcal{O}_{\widetilde{X}}} \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^{\vee})\big)\\ \to \tilde{\imath}^{\,*}\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{\,-1}\mathcal{O}_{\widetilde{X}}} \tilde{\imath}^{\,-1}\big(\bigwedge^{d-1} \mathcal{T}_{\widetilde{X}/X} \otimes_{\mathcal{O}_{\widetilde{X}}} \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^{\vee})\big)\end{aligned}$$

is given by

$$\begin{aligned}(4.2)\quad & \partial(D \otimes \xi_1 \wedge \cdots \wedge \xi_d \otimes v)\\ := & \sum_i (-1)^{i+1}\big(D\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_d \otimes v\\ & \qquad\qquad - D \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_d \otimes \xi_i v\big)\\ + & \sum_{i<j} (-1)^{i+j}\big(D \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_d \otimes v\big).\end{aligned}$$

for $D \in \tilde{\imath}^{\,*}\mathcal{D}_{\widetilde{X}}$, $\xi, \cdots, \xi_d \in \mathcal{T}_{\widetilde{X}/X}$, and $v \in \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^{\vee})$. In view of the proof of Lemma 3.4, we have only to see that the pull-back $(i_o)^*$ sends the complex

$$\tilde{\imath}^{\,*}\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{\,-1}\mathcal{O}_{\widetilde{X}}} \tilde{\imath}^{\,-1}\big(\bigwedge^{\bullet} \mathcal{T}_{\widetilde{X}/X} \otimes_{\mathcal{O}_{\widetilde{X}}} \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^{\vee})\big)$$

to $U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} (\bigwedge^{\bullet} \bar{\mathfrak{u}} \otimes V \otimes \mathbb{C}_{2\rho(\mathfrak{u})})$.

Write $V^d := \bigwedge^d \bar{\mathfrak{u}} \otimes V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}$ for simplicity. Since $\bigwedge^d \mathcal{T}_{\widetilde{X}/X} \otimes_{\mathcal{O}_{\widetilde{X}}} \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^{\vee})$ is isomorphic to the $\mathcal{O}_{\widetilde{X}}$-module $\mathcal{V}^d_{\widetilde{X}}$ associated with the L-module V^d, it follows that

$$\begin{aligned}&(i_o)^*\Big(\tilde{\imath}^{\,*}\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{\,-1}\mathcal{O}_{\widetilde{X}}} \tilde{\imath}^{\,-1}\big(\bigwedge^{d} \mathcal{T}_{\widetilde{X}/X} \otimes_{\mathcal{O}_{\widetilde{X}}} \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^{\vee})\big)\Big)\\ \simeq\ & (i_o)^*\big(\tilde{\imath}^{\,*}\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{\,-1}\mathcal{O}_{\widetilde{X}}} \tilde{\imath}^{\,-1}\mathcal{V}^d_{\widetilde{X}}\big)\\ \simeq\ & U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^d\end{aligned}$$

as in the proof of Lemma 3.4. Therefore, $\tilde{\imath}^{\,*}\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{\,-1}\mathcal{O}_{\widetilde{X}}} \tilde{\imath}^{\,-1}\mathcal{V}^d_{\widetilde{X}}$ is isomorphic to the K-equivariant $\mathcal{O}_{\widetilde{Y}}$-module associated with the $(L \cap K)$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^d$. Via this isomorphism, we view a section

$$f \in \tilde{\imath}^{\,*}\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{\,-1}\mathcal{O}_{\widetilde{X}}} \tilde{\imath}^{\,-1}\mathcal{V}^d_{\widetilde{X}}$$

as a regular function on an open set of K that takes values in $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^d$. Write $f(e) \in U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^d$ for the evaluation at the identity $e \in K$. The boundary map (4.2) is $\mathcal{O}_{\widetilde{Y}}$-linear and hence induces an operator

$$\partial_e : U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^d \to U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^{d-1}$$

such that $\partial_e(f(e)) = (\partial f)(e)$ for every $f \in \tilde{\imath}^{\,*}\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{\,-1}\mathcal{O}_{\widetilde{X}}} \tilde{\imath}^{\,-1}\mathcal{V}^d_{\widetilde{X}}$. It is enough to show that $\partial_e = \partial'$ on $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^{\bullet}$.

Put $Z := (\overline{U} \cdot L)/L \subset G/L = \widetilde{X}$ and write $i_Z : Z \to \widetilde{X}$ for the inclusion map so that $i_Z(Z) = \pi^{-1}(\{o\})$. Then under the isomorphism $Z \simeq \overline{U}$, there is a canonical isomorphism of $\overline{U}$-equivariant $\mathcal{O}$-modules $\iota : i_Z^*\mathcal{T}_{\widetilde{X}/X} \simeq \mathcal{T}_{\overline{U}}$.

For $\xi_1, \ldots, \xi_d \in \bar{\mathfrak{u}}$ and $v \in V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}$, put

$$m := \xi_1 \wedge \cdots \wedge \xi_d \otimes v \ \in\ V^d.$$

We will choose sections $\widetilde{\xi}_i \in \mathcal{T}_{\widetilde{X}/X}$ and $\widetilde{v} \in \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^\vee)$ on a neighborhood of the base point $o \in \widetilde{X}$ in the following way. Take $\widetilde{\xi}_i \in \mathcal{T}_{\widetilde{X}/X}$ such that $\widetilde{\xi}_i|_Z \in i_Z^*\mathcal{T}_{\widetilde{X}/X}$ corresponds to $(\xi_i)^R_{\overline{U}}$ under ι. The G-equivariant $\mathcal{O}_X$-module $\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^\vee$ is isomorphic to the $\mathcal{O}_X$-module associated with the $\overline{Q}$-module $V \otimes \mathbb{C}_{2\rho(\mathfrak{u})}$. Hence $f \in \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^\vee)$ is identified with a $(V \otimes \mathbb{C}_{2\rho(\mathfrak{u})})$-valued regular function on an open set of $\widetilde{X}$ satisfying $f(gq) = q^{-1} \cdot f(g)$ for $g \in G$ and $q \in \overline{Q}$. With this identification, we take a section $\widetilde{v}' \in \mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^\vee$ on a neighborhood of o such that $\widetilde{v}'(e) = v$. Define the section $\widetilde{v} \in \pi^*(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^\vee)$ as

$$\widetilde{v} := 1 \otimes \widetilde{v}' \in \mathcal{O}_{\widetilde{X}} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}(\mathcal{V}_X \otimes_{\mathcal{O}_X} \Omega_X^\vee).$$

and define the section $\widetilde{m} \in \mathcal{V}^d_{\widetilde{X}}$ in a neighborhood of o as

$$\widetilde{m} := \widetilde{\xi}_1 \wedge \cdots \wedge \widetilde{\xi}_d \otimes \widetilde{v} \in \mathcal{V}^d_{\widetilde{X}}.$$

Then

$$1 \otimes \widetilde{m} \in \tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \tilde{\imath}^{-1}\mathcal{V}^d_{\widetilde{X}}$$

satisfies $(1 \otimes \widetilde{m})(e) = 1 \otimes m$.

We have

$$\begin{aligned}
&\partial(1 \otimes \widetilde{m}) \\
&= \sum_i (-1)^{i+1} \Big((\xi_i)_{\widetilde{X}} \otimes \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widetilde{\xi}_d \otimes \widetilde{v} \\
&\qquad\qquad - 1 \otimes \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widetilde{\xi}_d \otimes \widetilde{\xi}_i \widetilde{v} \Big) \\
&+ \sum_{i<j} (-1)^{i+j} \left(1 \otimes [\widetilde{\xi}_i, \widetilde{\xi}_j] \wedge \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widehat{\widetilde{\xi}_j} \wedge \cdots \wedge \widetilde{\xi}_d \otimes \widetilde{v}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\partial'(1 \otimes m) \\
&= \sum_i (-1)^{i+1} \big(\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_d \otimes v - 1 \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_d \otimes \xi_i v\big) \\
&+ \sum_{i<j} (-1)^{i+j} \left(1 \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_d \otimes v\right).
\end{aligned}$$

Since $\widetilde{\xi}_i|_Z$ corresponds to $(\xi_i)^R_{\overline{U}}$, the tangent vectors at the base point o of the vector fields $\widetilde{\xi}_i$ and $(\xi_i)_{\widetilde{X}}$ have the relation: $(\widetilde{\xi}_i)_o = -((\xi_i)_{\widetilde{X}})_o$. Recall that the $\mathfrak{g}$-actions on $\mathcal{T}_{\widetilde{X}/X}$ and $\pi^*(\mathcal{V}_{\widetilde{X}} \otimes \Omega_{\widetilde{X}})$ are defined as the differentials of the G-equivariant structures on them. Our choice implies that $\widetilde{\xi}_j|_Z$ is left $\overline{U}$-invariant and hence $\xi_i \cdot \widetilde{\xi}_j|_Z = 0$. We therefore have

$$(1 \otimes \xi_i(\widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widetilde{\xi}_d) \otimes \widetilde{v})(e) = 0.$$

In addition, our choice of $\widetilde{v}$ implies that $\mathcal{T}_{\widetilde{X}/X}\widetilde{v} = 0$ and $(\xi_i\widetilde{v})(e) = \xi_i v$. As a result,

$$\begin{aligned}
&\Big((\xi_i)_{\widetilde{X}} \otimes \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widetilde{\xi}_d \otimes \widetilde{v} - 1 \otimes \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widetilde{\xi}_d \otimes \widetilde{\xi_i v}\Big)(e) \\
&= \Big((\xi_i)_{\widetilde{X}} \otimes \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widetilde{\xi}_d \otimes \widetilde{v}\Big)(e) \\
&= \big(\xi_i(1 \otimes \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widetilde{\xi}_d \otimes \widetilde{v})\big)(e) - (1 \otimes \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widetilde{\xi}_d \otimes \xi_i\widetilde{v})(e) \\
&= \xi_i\big((1 \otimes \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widetilde{\xi}_d \otimes \widetilde{v})(e)\big) - (1 \otimes \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widetilde{\xi}_d \otimes \xi_i\widetilde{v})(e) \\
&= \xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi}_i \wedge \cdots \wedge \xi_d \otimes v - 1 \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi}_i \wedge \cdots \wedge \xi_d \otimes \xi_i v.
\end{aligned}$$

Moreover, $[\widetilde{\xi}_i, \widetilde{\xi}_j]|_Z$ corresponds to $[(\xi_i)\frac{R}{U}, (\xi_j)\frac{R}{U}] = ([\xi_i, \xi_j])\frac{R}{U}$. Hence

$$\begin{aligned}
&\big(1 \otimes [\widetilde{\xi}_i, \widetilde{\xi}_j] \wedge \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widehat{\widetilde{\xi}_j} \wedge \cdots \wedge \widetilde{\xi}_d \otimes \widetilde{v}\big)(e) \\
&= 1 \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi}_i \wedge \cdots \wedge \widehat{\xi}_j \wedge \cdots \wedge \xi_d \otimes v.
\end{aligned}$$

We thus conclude that

$$\partial_e(1 \otimes m) = \partial_e((1 \otimes \widetilde{m})(e)) = (\partial(1 \otimes \widetilde{m}))(e) = \partial'(1 \otimes m).$$

Since ∂_e and ∂' commute with $\mathfrak{g}$-actions, $\partial_e = \partial'$. Therefore, we obtain an isomorphism (4.1) and prove the theorem. $\square$

5. Construction of Parabolic Subalgebras

Let G_0 be a connected real linear reductive Lie group with Lie algebra $\mathfrak{g}_0$ and σ an involution of G_0. Let G'_0 be the identity component of the fixed point set G_0^σ. There exists a Cartan involution θ of G_0 that commutes with σ. The corresponding maximal compact subgroups of G_0 and G'_0 are written as $K_0 := G_0^\theta$ and $K'_0 := (G'_0)^\theta$, respectively. The Cartan decompositions are written as $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ and $\mathfrak{g}'_0 = \mathfrak{k}'_0 + \mathfrak{p}'_0$. We denote by $\mathfrak{g}, \mathfrak{g}', \mathfrak{k}$, etc. the complexifications of $\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{k}_0$, etc. Let σ and θ also denote the induced actions on $\mathfrak{g}_0$ and their complex linear extensions to $\mathfrak{g}$.

Definition 5.1. Let V be a $(\mathfrak{g}', K')$-module. We say that V is *discretely decomposable* if V admits a filtration $\{V_p\}_{p\in\mathbb{N}}$ such that $V = \bigcup_{p\in\mathbb{N}} V_p$ and V_p is of finite length as a $(\mathfrak{g}', K')$-module for each $p \in \mathbb{N}$.

If V is unitarizable and discretely decomposable, then V is an algebraic direct sum of irreducible $(\mathfrak{g}', K')$-modules (see [12, Lemma 1.3]).

Definition 5.2. Suppose that $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$. We say that $\mathfrak{q}$ is *σ-open* if $\mathfrak{q} \cap \mathfrak{k} + \mathfrak{k}' = \mathfrak{k}$.

Remark 5.3. If $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$, there exists a σ-open θ-stable parabolic subalgebra that is conjugate to $\mathfrak{q}$ under the adjoint action of K_0.

We write $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}'}$ for the nilpotent cones of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Let $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}$ denote the projection from $\mathfrak{g}$ onto $\mathfrak{g}'$ along $\mathfrak{g}^{-\sigma}$.

Theorem 5.4. *Let (G_0, G'_0) be a symmetric pair of connected real linear reductive Lie groups defined by an involution σ. Let $\mathfrak{q}$ be a σ-open θ-stable parabolic subalgebra of $\mathfrak{g}$. Then the following three conditions are equivalent.*

(i) $A_{\mathfrak{q}}(\lambda)$ *is nonzero and discretely decomposable as a* $(\mathfrak{g}', K')$*-module for some* λ *in the weakly fair range.*
(ii) $A_{\mathfrak{q}}(\lambda)$ *is discretely decomposable as a* $(\mathfrak{g}', K')$*-module for any* λ *in the weakly fair range.*
(iii) *Put* $\mathfrak{q}' := N_{\mathfrak{k}'}(\mathfrak{q} \cap \mathfrak{p}') + (\mathfrak{q} \cap \mathfrak{p}')$, *where* $N_{\mathfrak{k}'}(\mathfrak{q} \cap \mathfrak{p}')$ *is the normalizer of* $\mathfrak{q} \cap \mathfrak{p}'$ *in* $\mathfrak{k}'$. *Then* $\mathfrak{q}'$ *is a* θ*-stable parabolic subalgebra of* $\mathfrak{g}'$.

The proof is based on the following criterion for the discrete decomposability ([12, Theorem 4.2]).

Fact 5.5. *In the setting of Theorem 5.4, the following conditions are equivalent.*

(i) $A_{\mathfrak{q}}(\lambda)$ *is nonzero and discretely decomposable as a* $(\mathfrak{g}', K')$*-module for some* λ *in the weakly fair range.*
(ii) $A_{\mathfrak{q}}(\lambda)$ *is discretely decomposable as a* $(\mathfrak{g}', K')$*-module for any* λ *in the weakly fair range.*
(iv) $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathfrak{u} \cap \mathfrak{p}) \subset \mathcal{N}_{\mathfrak{g}'}$ *for the nilradical* $\mathfrak{u}$ *of* $\mathfrak{q}$.

We use the following lemma for the proof of Theorem 5.4.

Lemma 5.6. *Let* V *be a finite-dimensional vector space with a non-degenerate symmetric bilinear form. For subspaces* $V_1 \subset V_2 \subset V$, *we denote by* $V_1^{\perp V_2}$ *the set of all vectors in* V_2 *that are orthogonal to* V_1.

Suppose that X *is a subspace of* V *such that* $V = X \oplus X^{\perp V}$. *Let* p *be the projection onto* X *along* $X^{\perp V}$. *Then for any subspace* $W \subset V$, *it follows that*

$$(W \cap X)^{\perp X} = p(W^{\perp V}).$$

Proof. We have

$$(W \cap X)^{\perp X} = (W \cap X)^{\perp V} \cap X = (W^{\perp V} + X^{\perp V}) \cap X = p(W^{\perp V}),$$

so the assertion is verified. □

Proof of Theorem 5.4. First of all, $\mathfrak{q}'$ defined in (iii) is a subalgebra of $\mathfrak{g}$ because $[\mathfrak{q} \cap \mathfrak{p}', \mathfrak{q} \cap \mathfrak{p}'] \subset \mathfrak{q} \cap \mathfrak{k}' \subset N_{\mathfrak{k}'}(\mathfrak{q} \cap \mathfrak{p}')$.

Choose an invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that the subspaces $\mathfrak{k}', \mathfrak{k}^{-\sigma}, \mathfrak{p}'$, and $\mathfrak{p}^{-\sigma}$ are mutually orthogonal. We use the letter $^{\perp}$ for orthogonal spaces with respect to $\langle \cdot, \cdot \rangle$ as in Lemma 5.6.

It is enough to prove the equivalence of (iii) and (iv) by Fact 5.5.

Assume that (iii) holds. The subspaces $\mathfrak{u} = \mathfrak{q}^{\perp \mathfrak{g}}$ and $\mathfrak{u}' = \mathfrak{q}'^{\perp \mathfrak{g}'}$ are the nilradicals of $\mathfrak{q}$ and $\mathfrak{q}'$, respectively. Because $\mathfrak{q}$ and $\mathfrak{q}'$ are θ-stable, we have $(\mathfrak{q} \cap \mathfrak{p})^{\perp \mathfrak{p}} = \mathfrak{u} \cap \mathfrak{p}$ and $(\mathfrak{q}' \cap \mathfrak{p}')^{\perp \mathfrak{p}'} = \mathfrak{u}' \cap \mathfrak{p}'$. In view of Lemma 5.6 and $\mathfrak{q} \cap \mathfrak{p}' = \mathfrak{q}' \cap \mathfrak{p}'$, we get

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathfrak{u} \cap \mathfrak{p}) = \mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}((\mathfrak{q} \cap \mathfrak{p})^{\perp \mathfrak{p}}) = (\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'} = (\mathfrak{q}' \cap \mathfrak{p}')^{\perp \mathfrak{p}'} = \mathfrak{u}' \cap \mathfrak{p}'.$$

The right side is contained in $\mathcal{N}_{\mathfrak{g}'}$. This shows (iv).

Assume that (iv) holds. As we have seen above,

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}((\mathfrak{q} \cap \mathfrak{p})^{\perp \mathfrak{p}}) = (\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'}.$$

Since the vector space $(\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'}$ is contained in the nilpotent cone of $\mathfrak{g}'$, the bilinear form $\langle \cdot, \cdot \rangle$ is zero on $(\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'}$ and hence $(\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'} \subset \mathfrak{q} \cap \mathfrak{p}'$. Then it follows that

$N_{\mathfrak{k}'}(\mathfrak{q}\cap\mathfrak{p}') = [(\mathfrak{q}\cap\mathfrak{p}'),(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'}]^{\perp\mathfrak{k}'}$. Indeed, for $x\in\mathfrak{k}'$,

$$\begin{aligned} x\in[(\mathfrak{q}\cap\mathfrak{p}'),(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'}]^{\perp\mathfrak{k}'} &\Leftrightarrow \langle x,[(\mathfrak{q}\cap\mathfrak{p}'),(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'}]\rangle=\{0\}\\ &\Leftrightarrow \langle [x,(\mathfrak{q}\cap\mathfrak{p}')],(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'}\rangle=\{0\}\\ &\Leftrightarrow [x,(\mathfrak{q}\cap\mathfrak{p}')]\in\mathfrak{q}\cap\mathfrak{p}'\\ &\Leftrightarrow x\in N_{\mathfrak{k}'}(\mathfrak{q}\cap\mathfrak{p}'). \end{aligned}$$

Put $\mathfrak{q}' := N_{\mathfrak{k}'}(\mathfrak{q}\cap\mathfrak{p}')+(\mathfrak{q}\cap\mathfrak{p}')$. Then

$$\mathfrak{q}'^{\perp\mathfrak{g}'} = N_{\mathfrak{k}'}(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{k}'}+(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'} = [(\mathfrak{q}\cap\mathfrak{p}'),(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'}]+(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'}.$$

Since $[(\mathfrak{q}\cap\mathfrak{p}'),(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'}]\subset[(\mathfrak{q}\cap\mathfrak{p}'),(\mathfrak{q}\cap\mathfrak{p}')]\subset N_{\mathfrak{k}'}(\mathfrak{q}\cap\mathfrak{p}')$, we see that $\mathfrak{q}'^{\perp\mathfrak{g}'}\subset\mathfrak{q}'$. We therefore have $\langle x,y\rangle=0$ for $x,y\in\mathfrak{q}'^{\perp\mathfrak{g}'}$. Moreover, $\mathfrak{q}'^{\perp\mathfrak{g}'}$ is a subalgebra of $\mathfrak{g}'$ because

$$\langle[\mathfrak{q}'^{\perp\mathfrak{g}'},\mathfrak{q}'^{\perp\mathfrak{g}'}],\mathfrak{q}'\rangle = \langle\mathfrak{q}'^{\perp\mathfrak{g}'},[\mathfrak{q}'^{\perp\mathfrak{g}'},\mathfrak{q}']\rangle\subset\langle\mathfrak{q}'^{\perp\mathfrak{g}'},\mathfrak{q}'\rangle=\{0\}.$$

As a consequence, $\mathfrak{q}'^{\perp\mathfrak{g}'}$ is a solvable Lie algebra and hence contained in some Borel subalgebra $\mathfrak{b}'$ of $\mathfrak{g}'$. Write $\mathfrak{n}'$ for the nilradical of $\mathfrak{b}'$ so $\mathfrak{n}'=\mathfrak{b}'^{\perp\mathfrak{g}'}$. Let $M:=N_{K'}(\mathfrak{q}\cap\mathfrak{p}')$ be the normalizer of $\mathfrak{q}\cap\mathfrak{p}'$, which is an algebraic subgroup of K'. Then M has a Levi decomposition with reductive part M_R and unipotent part M_U (see [6, §VIII.4]). If we denote by $\mathfrak{m}_R$ and $\mathfrak{m}_U$ the Lie algebras of M_R and M_U, respectively, then the bilinear form $\langle\cdot,\cdot\rangle$ is non-degenerate on $\mathfrak{m}_R$ and zero on $\mathfrak{m}_U$. We then conclude that the nilradical of $N_{\mathfrak{k}'}(\mathfrak{q}\cap\mathfrak{p}')$ equals the radical of $N_{\mathfrak{k}'}(\mathfrak{q}\cap\mathfrak{p}')$ with respect to the bilinear form. As a result, $[(\mathfrak{q}\cap\mathfrak{p}'),(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'}]=N_{\mathfrak{k}'}(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{k}'}$ is the nilradical of $N_{\mathfrak{k}'}(\mathfrak{q}\cap\mathfrak{p}')$ and hence $[(\mathfrak{q}\cap\mathfrak{p}'),(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'}]\subset\mathfrak{n}'$. Since $(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'}\subset\mathcal{N}_{\mathfrak{g}'}\cap\mathfrak{b}'=\mathfrak{n}'$, it follows that $\mathfrak{q}'^{\perp\mathfrak{g}'}=[(\mathfrak{q}\cap\mathfrak{p}'),(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'}]+(\mathfrak{q}\cap\mathfrak{p}')^{\perp\mathfrak{p}'}\subset\mathfrak{n}'$. Hence we see that $\mathfrak{q}'\supset\mathfrak{n}'^{\perp\mathfrak{g}'}=\mathfrak{b}'$ and $\mathfrak{q}'$ is a parabolic subalgebra of $\mathfrak{g}'$, showing (iii). □

Retain the notation and the assumption of Theorem 5.4 and suppose that the equivalent conditions in Theorem 5.4 are satisfied. Let $\mathcal{Q}$ be the set of all θ-stable parabolic subalgebras $\mathfrak{q}'_i$ of $\mathfrak{g}'$ such that $\mathfrak{q}'_i\cap\mathfrak{p}'=\mathfrak{q}\cap\mathfrak{p}'$. Then the parabolic subalgebra $\mathfrak{q}'=N_{\mathfrak{k}'}(\mathfrak{q}\cap\mathfrak{p}')+(\mathfrak{q}\cap\mathfrak{p}')$ given in Theorem 5.4 is a unique maximal element of $\mathcal{Q}$.

On the other hand, a minimal element $\mathfrak{q}''$ of $\mathcal{Q}$ is constructed as follows. For the parabolic subalgebra $\mathfrak{q}'$ defined above, put $\mathfrak{l}'=\mathfrak{q}'\cap\overline{\mathfrak{q}'}$, which is a Levi part of $\mathfrak{q}'$. The θ-stable reductive subalgebra $\mathfrak{l}'$ decomposes as

$$\mathfrak{l}'=\bigoplus_{i\in I}\mathfrak{l}'_i\oplus\mathfrak{z}(\mathfrak{l}'),$$

where $\mathfrak{l}'_i$ are simple Lie algebras and $\mathfrak{z}(\mathfrak{l}')$ is the center of $\mathfrak{l}'$. Put $I_c:=\{i\in I:\mathfrak{l}'_i\subset\mathfrak{k}'\}$ and define

$$\mathfrak{l}'_c:=\bigoplus_{i\in I_c}\mathfrak{l}'_i\oplus(\mathfrak{z}(\mathfrak{l}')\cap\mathfrak{k}'),\quad \mathfrak{l}'_n:=\bigoplus_{i\notin I_c}\mathfrak{l}'_i\oplus(\mathfrak{z}(\mathfrak{l}')\cap\mathfrak{p}').$$

Then we have

$$\mathfrak{l}'=\mathfrak{l}'_c\oplus\mathfrak{l}'_n,\quad \mathfrak{l}'_n=[(\mathfrak{l}'\cap\mathfrak{p}'),(\mathfrak{l}'\cap\mathfrak{p}')]+\mathfrak{l}'\cap\mathfrak{p}',\quad \mathfrak{l}'_c\subset\mathfrak{k}'.$$

Take a Borel subalgebra $\mathfrak{b}(\mathfrak{l}'_c)$ of $\mathfrak{l}'_c$ and define

$$\mathfrak{q}'':=\mathfrak{b}(\mathfrak{l}'_c)\oplus\mathfrak{l}'_n\oplus\mathfrak{u}'. \tag{5.1}$$

We claim that $\mathfrak{q}''$ is a minimal element of $\mathcal{Q}$ and every minimal element is obtained in this way. Indeed, since any element $\mathfrak{q}'_i$ of $\mathcal{Q}$ is contained in $\mathfrak{q}'$, the parabolic subalgebra $\mathfrak{q}'_i$ decomposes as $(\mathfrak{q}'_i \cap \mathfrak{l}') \oplus \mathfrak{u}'$. The condition $\mathfrak{q}'_i \cap \mathfrak{p}' = \mathfrak{q} \cap \mathfrak{p}'$ implies that $\mathfrak{q}'_i \supset \mathfrak{l}' \cap \mathfrak{p}'$ and hence $\mathfrak{q}'_i \supset \mathfrak{l}'_n$. As a consequence, the set $\mathcal{Q}$ consists of the Lie algebras $\mathfrak{q}(\mathfrak{l}'_c) \oplus \mathfrak{l}'_n \oplus \mathfrak{u}'$ for parabolic subalgebras $\mathfrak{q}(\mathfrak{l}'_c)$ of $\mathfrak{l}'_c$. Our claim follows from this. In particular, a minimal element of $\mathcal{Q}$ is unique up to inner automorphisms of $\mathfrak{l}'_c$.

We note here some observations on Lie algebras for later use.

Lemma 5.7. *Retain the notation and the assumption above. Then*

$$\mathfrak{q} \cap \mathfrak{g}' = (\mathfrak{q} \cap \mathfrak{l}'_c) \oplus \mathfrak{l}'_n \oplus \mathfrak{u}',$$

and

$$[(\mathfrak{l}'_n + \mathfrak{u}'), \mathfrak{g}] \subset \mathfrak{q} + \mathfrak{g}'.$$

Proof. From $\mathfrak{q} \cap \mathfrak{k}' \subset N_{\mathfrak{k}'}(\mathfrak{q} \cap \mathfrak{p}')$ and $\mathfrak{q} \cap \mathfrak{p}' = \mathfrak{q}' \cap \mathfrak{p}'$, we have $\mathfrak{q} \cap \mathfrak{g}' \subset \mathfrak{q}'$. From the proof of Theorem 5.4, we have

$$\begin{aligned}\mathfrak{u}' = \mathfrak{q}'^{\perp \mathfrak{g}'} &= [(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'}] + (\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'} \\ &\subset [(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')] + (\mathfrak{q} \cap \mathfrak{p}') \subset \mathfrak{q} \cap \mathfrak{g}'.\end{aligned}$$

Moreover, $\mathfrak{l}'_n = [(\mathfrak{l}' \cap \mathfrak{p}'), (\mathfrak{l}' \cap \mathfrak{p}')] + (\mathfrak{l}' \cap \mathfrak{p}')$ and $\mathfrak{l}' \cap \mathfrak{p}' \subset \mathfrak{q}' \cap \mathfrak{p}' = \mathfrak{q} \cap \mathfrak{p}'$ imply that $\mathfrak{l}'_n \subset \mathfrak{q} \cap \mathfrak{g}'$. Hence $\mathfrak{q} \cap \mathfrak{g}'$ decomposes as $\mathfrak{q} \cap \mathfrak{g}' = (\mathfrak{q} \cap \mathfrak{l}'_c) \oplus \mathfrak{l}'_n \oplus \mathfrak{u}'$.

For the second assertion, we see that $[(\mathfrak{q} \cap \mathfrak{p}'), \mathfrak{g}] \subset \mathfrak{q} + \mathfrak{g}'$. Indeed, the assumption $(\mathfrak{q} \cap \mathfrak{k}) + \mathfrak{k}' = \mathfrak{k}$ implies that

$$[(\mathfrak{q} \cap \mathfrak{p}'), \mathfrak{k}] = [(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{k})] + [(\mathfrak{q} \cap \mathfrak{p}'), \mathfrak{k}'] \subset \mathfrak{q} + \mathfrak{g}'$$

and $[(\mathfrak{q} \cap \mathfrak{p}'), \mathfrak{p}] \subset \mathfrak{k} \subset \mathfrak{q} + \mathfrak{g}'$. Hence $[(\mathfrak{q} \cap \mathfrak{p}'), \mathfrak{g}] \subset \mathfrak{q} + \mathfrak{g}'$. Then the inclusion $[\mathfrak{u}', \mathfrak{g}] \subset \mathfrak{q} + \mathfrak{g}'$ follows from $\mathfrak{u}' \subset [(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')] + (\mathfrak{q} \cap \mathfrak{p}')$, and the inclusion $[\mathfrak{l}'_n, \mathfrak{g}] \subset \mathfrak{q} + \mathfrak{g}'$ follows from $\mathfrak{l}'_n = [(\mathfrak{l}' \cap \mathfrak{p}'), (\mathfrak{l}' \cap \mathfrak{p}')] + (\mathfrak{l}' \cap \mathfrak{p}')$. □

6. Upper Bound on Branching Law

We retain the notation of the previous section.

Proposition 6.1. *Suppose that the equivalent conditions in Theorem 5.4 hold for a σ-open θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. Define $\mathfrak{q}'$ as in Theorem 5.4 and define $\overline{Q'}$ as the parabolic subgroup of G' with Lie algebra $\overline{\mathfrak{q}'}$. Then $\overline{Q} \cap G' \subset \overline{Q'}$. Moreover, the following is a Cartesian square.*

$$\begin{tikzcd} K'/(\overline{Q} \cap K') \arrow[r, "i^o"] \arrow[d] & G'/(\overline{Q} \cap G') \arrow[d, "\pi"] \\ K'/(\overline{Q'} \cap K') \arrow[r, "i'"] & G'/\overline{Q'} \end{tikzcd}$$

In particular, i^o is a closed immersion.

Proof. Let $g \in \overline{Q} \cap G'$. To see $g \in \overline{Q'}$, it enough to show that $\mathrm{Ad}(g)$ normalizes $\overline{\mathfrak{q}'}$ because $\overline{Q'}$ is self-normalizing. By Lemma 5.7, $\overline{\mathfrak{u}'} \subset \bar{\mathfrak{q}} \cap \mathfrak{g}' \subset \overline{\mathfrak{q}'}$. Therefore, $\mathrm{Ad}(g)(\bar{\mathfrak{q}} \cap \mathfrak{g}') = \bar{\mathfrak{q}} \cap \mathfrak{g}'$ implies that $\mathrm{Ad}(g)\overline{\mathfrak{u}'} \subset \overline{\mathfrak{q}'}$. Then $\mathrm{Ad}(g)\overline{\mathfrak{q}'} \subset \overline{\mathfrak{q}'}$ follows from the lemma below:

Lemma 6.2. *Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{q}$ a parabolic subalgebra. If $\phi(\mathfrak{u}) \subset \mathfrak{q}$ for the nilradical $\mathfrak{u}$ of $\mathfrak{q}$ and an inner automorphism $\phi \in \mathrm{Int}(\mathfrak{g})$, then $\phi(\mathfrak{q}) = \mathfrak{q}$.*

Proof. There exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in both $\mathfrak{q}$ and $\phi(\mathfrak{q})$. Our assumption amounts to the inclusion of the sets of $\mathfrak{h}$-roots $\Delta(\phi(\mathfrak{u}), \mathfrak{h}) \subset \Delta(\mathfrak{q}, \mathfrak{h})$. Write $\mathfrak{l}$ for the Levi part of $\mathfrak{q}$ containing $\mathfrak{h}$. Then

$$\Delta(\phi(\mathfrak{q}), \mathfrak{h}) \cap \Delta(\mathfrak{q}, \mathfrak{h}) = \Delta(\phi(\mathfrak{u}), \mathfrak{h}) \cup (\Delta(\phi(\mathfrak{l}), \mathfrak{h}) \cap \Delta(\mathfrak{q}, \mathfrak{h})).$$

As a result, $\phi(\mathfrak{q}) \cap \mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$. In particular, $\phi(\mathfrak{q})$ and $\mathfrak{q}$ have a common Borel subalgebra. Since ϕ is inner, this implies that $\phi(\mathfrak{q}) = \mathfrak{q}$. □

Returning to the proof of Proposition 6.1, we now prove that the diagram is a Cartesian square. This is equivalent to that $\overline{Q'} = (\overline{Q'} \cap K') \cdot (\overline{Q} \cap G')$. The inclusion $\overline{Q'} \supset (\overline{Q'} \cap K') \cdot (\overline{Q} \cap G')$ follows from $\overline{Q'} \supset (\overline{Q} \cap G')$. Since $\overline{Q'}$ is connected and θ-stable, it is generated by $\overline{Q'} \cap K'$ and $\exp(\overline{\mathfrak{q}'} \cap \mathfrak{p}')$ as a group. For $k \in \overline{Q'} \cap K'$ and $x \in \overline{\mathfrak{q}'} \cap \mathfrak{p}'$, we have $\exp(x)k = k \exp(\mathrm{Ad}(k^{-1})x)$ and $\mathrm{Ad}(k^{-1})x \in \overline{\mathfrak{q}'} \cap \mathfrak{p}'$. Using this equation iteratively, we can write any element of $\overline{Q'}$ as $k \exp(x_1) \cdots \exp(x_n)$ for $k \in \overline{Q'} \cap K'$ and $x_1, \ldots, x_n \in \overline{\mathfrak{q}'} \cap \mathfrak{p}'$. Then $\overline{\mathfrak{q}'} \cap \mathfrak{p}' = \bar{\mathfrak{q}} \cap \mathfrak{p}'$ implies that $\exp(x_1) \cdots \exp(x_n) \in \overline{Q} \cap G'$. Hence $\overline{Q'} \subset (\overline{Q'} \cap K') \cdot (\overline{Q} \cap G')$ as required. □

Now we consider the restriction $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$. We assume that λ is linear, so the $(\mathfrak{l}, L \cap K)$-action on $\mathbb{C}_\lambda$ can be uniquely extended to an L-action or a $\overline{Q}$-action.

Define

$$V^p := \bigwedge^{\mathrm{top}} (\mathfrak{g}/(\bar{\mathfrak{q}} + \mathfrak{g}')) \otimes S^p(\mathfrak{g}/(\bar{\mathfrak{q}} + \mathfrak{g}'))$$

regarded as a $(\overline{Q} \cap G')$-module by the adjoint action and define

$$W^p := \mathrm{Ind}_{\overline{Q} \cap G'}^{\overline{Q'}} (\mathbb{C}_\lambda|_{\overline{Q} \cap G'} \otimes V^p).$$

By Lemma 5.7, the unipotent radical $\overline{U'}$ of $\overline{Q'}$ is contained in $\overline{Q} \cap G'$ and $\overline{U'}$ acts trivially on $\mathbb{C}_\lambda|_{\overline{Q} \cap G'} \otimes V^p$. Therefore, $\overline{U'}$ acts trivially on W^p. Then W^p is written as a direct sum of irreducible finite-dimensional $\overline{Q'}$-modules and $\overline{U'}$ acts trivially on all the irreducible components. As an L'-module, we have

$$W^p \simeq \mathrm{Ind}_{\overline{Q} \cap L'}^{L'} (\mathbb{C}_\lambda|_{\overline{Q} \cap L'} \otimes V^p).$$

Theorem 6.3. *Let (G_0, G_0') be a symmetric pair of connected real linear reductive Lie groups defined by an involution σ. Let $\mathfrak{q}$ be a σ-open θ-stable parabolic subalgebra of $\mathfrak{g}$. Suppose that $A_{\mathfrak{q}}(\lambda)$ is nonzero and discretely decomposable as a $(\mathfrak{g}', K')$-module with λ linear, unitary, and in the weakly fair range. Define*

$$\mathfrak{q}' := N_{\mathfrak{k}'}(\mathfrak{q} \cap \mathfrak{p}') + (\mathfrak{q} \cap \mathfrak{p}'),$$

and

$$W^p := \mathrm{Ind}_{\overline{Q} \cap G'}^{\overline{Q'}} \left(\mathbb{C}_\lambda|_{\overline{Q} \cap G'} \otimes \bigwedge^{\mathrm{top}} (\mathfrak{g}/(\bar{\mathfrak{q}} + \mathfrak{g}')) \otimes S^p(\mathfrak{g}/(\bar{\mathfrak{q}} + \mathfrak{g}')) \right).$$

Then there exists an injective homomorphism of $(\mathfrak{g}', K')$-modules

$$A_{\mathfrak{q}}(\lambda) \to \bigoplus_{p=0}^{\infty} (\Pi_{L' \cap K'}^{K'})_{s'} \big(U(\mathfrak{g}') \otimes_{U(\overline{\mathfrak{q}'})} (W^p \otimes \mathbb{C}_{2\rho(\mathfrak{u}')}) \big) \tag{6.1}$$

for $s' = \dim(\mathfrak{u}' \cap \mathfrak{k}')$.

Proof. Suppose that $A_{\mathfrak{q}}(\lambda)$ is nonzero and discretely decomposable as a $(\mathfrak{g}', K')$-module with λ linear, unitary, and in the weakly fair range. Let $\overline{Q}$, G', and K' be the connected subgroups of G with Lie algebras $\bar{\mathfrak{q}}$, $\mathfrak{g}'$ and $\mathfrak{k}'$, respectively. We set

$$\begin{aligned} X &= G/\overline{Q}, \quad X^o = G'/(\overline{Q} \cap G'), \\ Y &= K/(\overline{Q} \cap K), \quad Y^o = K'/(\overline{Q} \cap K'), \end{aligned}$$

$$\begin{tikzcd} Y \arrow[r, "i"] & X \\ Y^o \arrow[u, "j_K"] \arrow[r, "i^o"] & X^o \arrow[u, "j"'] \end{tikzcd}$$

where the maps i^o, i, j, and j_K are the inclusion maps. The map j_K is an open immersion because $\mathfrak{q}$ is σ-open. By Lemma 6.1, i^o is a closed immersion and hence $i(Y) \cap j(X^o) = i(j_K(Y^o))$.

Let $\mathcal{L}_{\lambda,X}$ be the $\mathcal{O}_X$-module associated with the $\overline{Q}$-module $\mathbb{C}_\lambda$ as in Section 3. Then Theorem 4.1 says $\Gamma(X, i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{L}_{\lambda,X})$ is isomorphic to $A_{\mathfrak{q}}(\lambda)$ as a $(\mathfrak{g}, K)$-module. We see that

$$j^{-1} i_+ \mathcal{O}_Y \simeq j^{-1}(j \circ i^o)_+ \mathcal{O}_{Y^o} \simeq j^{-1} j_+ (i^o_+ \mathcal{O}_{Y^o}).$$

Let $\{F_p \mathcal{D}_X\}_{p \geq 0}$ be the filtration by normal degree with respect to j. This induces a filtration $\{F_p j^{-1} i_+ \mathcal{O}_Y\}$ on $j^{-1} i_+ \mathcal{O}_Y$ and a filtration $\{F_p j^{-1}(i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{L}_{\lambda,X})\}$ on $j^{-1}(i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{L}_{\lambda,X})$. Applying Lemma 3.3 for $\mathcal{M} = i^o_+ \mathcal{O}_{Y^o}$, we have isomorphisms of $\mathcal{O}_{X^o}$-modules

$$\begin{aligned} F_p j^{-1} i_+ \mathcal{O}_Y / F_{p-1} j^{-1} i_+ \mathcal{O}_Y &\simeq F_p j^{-1} j_+ (i^o_+ \mathcal{O}_{Y^o}) / F_{p-1} j^{-1} j_+ (i^o_+ \mathcal{O}_{Y^o}) \\ &\simeq (i^o_+ \mathcal{O}_{Y^o}) \otimes_{\mathcal{O}_{X^o}} \Omega^\vee_{X/X^o} \otimes_{\mathcal{O}_{X^o}} j^{-1}(\mathcal{I}^p_{X^o}/\mathcal{I}^{p+1}_{X^o})^\vee, \end{aligned}$$

which commute with the actions of $\mathfrak{g}'$ and K'. The G'-equivariant $\mathcal{O}_{X^o}$-module $\Omega^\vee_{X/X^o} \otimes_{\mathcal{O}_{X^o}} j^{-1}(\mathcal{I}^p_{X^o}/\mathcal{I}^{p+1}_{X^o})^\vee$ is isomorphic to the $\mathcal{O}_{X^o}$-module $\mathcal{V}^p_{X^o}$ associated with the $(\overline{Q'} \cap G')$-module

$$V^p := \bigwedge^{\mathrm{top}} (\mathfrak{g}/(\bar{\mathfrak{q}} + \mathfrak{g}')) \otimes S^p(\mathfrak{g}/(\bar{\mathfrak{q}} + \mathfrak{g}')).$$

We write $\mathcal{L}_{\lambda,X^o}$ for the $\mathcal{O}_{X^o}$-module associated with $\mathbb{C}_\lambda|_{\overline{Q} \cap G'}$. Then $j^*\mathcal{L}_{\lambda,X} \simeq \mathcal{L}_{\lambda,X^o}$. As a result, we get an isomorphism

$$\begin{aligned} &F_p j^{-1}(i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{L}_{\lambda,X}) / F_{p-1} j^{-1}(i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{L}_{\lambda,X}) \\ &\simeq i^o_+ \mathcal{O}_{Y^o} \otimes_{\mathcal{O}_{X^o}} \mathcal{L}_{\lambda,X^o} \otimes_{\mathcal{O}_{X^o}} \mathcal{V}^p_{X^o}. \end{aligned} \tag{6.2}$$

Since any section $m \in \Gamma(X, i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{L}_{\lambda,X})$ is K-finite, the support of m is Y unless $m = 0$. Therefore, the restriction map

$$r : \Gamma(X, i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{L}_{\lambda,X}) \to \Gamma(X^o, j^{-1}(i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{L}_{\lambda,X}))$$

is injective. Define the filtration $\{F_p A_{\mathfrak{q}}(\lambda)\}$ of the $(\mathfrak{g}', K')$-module $A_{\mathfrak{q}}(\lambda)$ by

$$F_p A_{\mathfrak{q}}(\lambda) := r^{-1} \Gamma(X^o, F_p j^{-1}(i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{L}_{\lambda,X}))$$

for

$$r : A_{\mathfrak{q}}(\lambda) \simeq \Gamma(X, i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{L}_{\lambda,X}) \to \Gamma(X^o, j^{-1}(i_+\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{L}_{\lambda,X})).$$

The induced map

$$\begin{aligned}&F_pA_{\mathfrak{q}}(\lambda)/F_{p-1}A_{\mathfrak{q}}(\lambda)\\ \to\ &\Gamma(X^o, F_pj^{-1}(i_+\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{L}_{\lambda,X}))\,/\,\Gamma(X^o, F_{p-1}j^{-1}(i_+\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{L}_{\lambda,X})).\end{aligned}$$

is injective. The unitarizability and the discrete decomposability of $A_{\mathfrak{q}}(\lambda)$ imply that there exists an isomorphism of the $(\mathfrak{g}', K')$-modules

$$A_{\mathfrak{q}}(\lambda)\simeq\bigoplus_{p=0}^{\infty}F_pA_{\mathfrak{q}}(\lambda)/F_{p-1}A_{\mathfrak{q}}(\lambda).$$

Consequently, we obtain injective maps of $(\mathfrak{g}', K')$-modules

$$\begin{aligned}(6.3)\quad &A_{\mathfrak{q}}(\lambda)\simeq\bigoplus_{p=0}^{\infty}F_pA_{\mathfrak{q}}(\lambda)/F_{p-1}A_{\mathfrak{q}}(\lambda)\\ &\to\bigoplus_{p=0}^{\infty}\Gamma(X^o, F_pj^{-1}(i_+\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{L}_{\lambda,X}))\,/\,\Gamma(X^o, F_{p-1}j^{-1}(i_+\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{L}_{\lambda,X}))\\ &\to\bigoplus_{p=0}^{\infty}\Gamma\big(X^o, F_pj^{-1}(i_+\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{L}_{\lambda,X})/F_{p-1}j^{-1}(i_+\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{L}_{\lambda,X})\big).\end{aligned}$$

The injectivity of the last map follows from the left exactness of the functor $\Gamma(X^o,\cdot)$.

We set

$$X'=G'/\overline{Q'},\quad Y'=K'/(\overline{Q'}\cap K'),$$

$$\begin{tikzcd} Y^o \arrow[r, "i^o"] \arrow[d, "\pi_K"'] & X^o \arrow[d, "\pi"] \\ Y' \arrow[r, "i'"] & X' \end{tikzcd}$$

where the maps in the commutative diagram are defined canonically. Since the diagram is a Cartesian square by Lemma 6.1 and π,π_K are smooth morphisms, the base change formula gives isomorphisms of $\mathcal{D}_{X^o}$-modules

$$i^o_+\mathcal{O}_{Y^o}\simeq i^o_+\pi_K^*\mathcal{O}_{Y'}\simeq\pi^*i'_+\mathcal{O}_{Y'}.$$

Then the projection formula gives the following isomorphisms of $\mathcal{O}_{X'}$-modules

$$\begin{aligned}\pi_*(i^o_+\mathcal{O}_{Y^o}\otimes_{\mathcal{O}_{X^o}}\mathcal{L}_{\lambda,X^o}\otimes_{\mathcal{O}_{X^o}}\mathcal{V}^p_{X^o})&\simeq\pi_*(\pi^*i'_+\mathcal{O}_{Y'}\otimes_{\mathcal{O}_{X^o}}\mathcal{L}_{\lambda,X^o}\otimes_{\mathcal{O}_{X^o}}\mathcal{V}^p_{X^o})\\ &\simeq i'_+\mathcal{O}_{Y'}\otimes_{\mathcal{O}_{X'}}\pi_*(\mathcal{L}_{\lambda,X^o}\otimes_{\mathcal{O}_{X^o}}\mathcal{V}^p_{X^o}),\end{aligned}$$

which commute with the actions of $\mathfrak{g}'$ and K'. Put $S:=\overline{Q'}/(\overline{Q}\cap G')$. By Lemma 3.2, $\pi_*(\mathcal{L}_{\lambda,X^o}\otimes_{\mathcal{O}_{X^o}}\mathcal{V}^p_{X^o})$ is isomorphic to the $\mathcal{O}_{X'}$-module $\mathcal{W}^p_{X'}$ associated with the $\overline{Q'}$-module $W^p:=\Gamma(S,\mathcal{V}^p_S)$, or equivalently

$$W^p:=\mathrm{Ind}_{\overline{Q}\cap G'}^{\overline{Q'}}\left(\mathbb{C}_\lambda|_{\overline{Q}\cap G'}\otimes\bigwedge^{\mathrm{top}}(\mathfrak{g}/(\bar{\mathfrak{q}}+\mathfrak{g}'))\otimes S^p(\mathfrak{g}/(\bar{\mathfrak{q}}+\mathfrak{g}'))\right).$$

Therefore,

$$\begin{aligned}(6.4)\qquad &\Gamma(X^o, i^o_+\mathcal{O}_{Y^o} \otimes_{\mathcal{O}_{X^o}} \mathcal{L}_{\lambda,X^o} \otimes_{\mathcal{O}_{X^o}} \mathcal{V}^p_{X^o})\\ &\simeq \Gamma(X', i'_+\mathcal{O}_{Y'} \otimes_{\mathcal{O}_{X'}} \pi_*(\mathcal{L}_{\lambda,X^o} \otimes_{\mathcal{O}_{X^o}} \mathcal{V}^p_{X^o}))\\ &\simeq \Gamma(X', i'_+\mathcal{O}_{Y'} \otimes_{\mathcal{O}_{X'}} \mathcal{W}^p_{X'}).\end{aligned}$$

Combining (6.2), (6.3), and (6.4), we obtain an injective $(\mathfrak{g}', K')$-homomorphism

$$A_{\mathfrak{q}}(\lambda) \to \bigoplus_{p=0}^{\infty} \Gamma(X', i'_+\mathcal{O}_{Y'} \otimes_{\mathcal{O}_{X'}} \mathcal{W}^p_{X'}). \tag{6.5}$$

Finally, Theorem 4.1 gives an isomorphism

$$\Gamma(X', i'_+\mathcal{O}_{Y'} \otimes_{\mathcal{O}_{X'}} \mathcal{W}^p_{X'}) \simeq (\Pi^{K'}_{L'\cap K'})_{s'}(U(\mathfrak{g}') \otimes_{U(\overline{\mathfrak{q}'})} (W^p \otimes \mathbb{C}_{2\rho(\mathfrak{u}')})),$$

so we have completed the proof. □

Let $\mathfrak{q}''$ be the θ-stable parabolic subalgebra of $\mathfrak{g}'$ defined by (5.1). In what follows, we show that the right side of (6.1) can be written as the direct sum of $(\mathfrak{g}', K')$-modules $A_{\mathfrak{q}''}(\lambda')$.

Let $L''_0 := N_{G'_0}(\overline{\mathfrak{q}''})$ be the normalizer of $\overline{\mathfrak{q}''}$ in G'_0. The complexified Lie algebra $\mathfrak{l}''$ decomposes as $\mathfrak{l}'' = (\mathfrak{l}'' \cap \mathfrak{l}'_c) \oplus \mathfrak{l}'_n$. Then $\mathfrak{h}'_c := \mathfrak{l}'' \cap \mathfrak{l}'_c$ is a Cartan subalgebra of $\mathfrak{l}'_c$. The center $\mathfrak{z}(\mathfrak{l}'')$ of $\mathfrak{l}''$ decomposes as

$$\mathfrak{z}(\mathfrak{l}'') = \mathfrak{h}'_c \oplus (\mathfrak{z}(\mathfrak{l}'') \cap \mathfrak{l}'_n).$$

Write $\lambda' = \lambda'_c + \lambda'_n$ for the corresponding decomposition of $\lambda' \in \mathfrak{z}(\mathfrak{l}'')^*$. We take $\Delta(\mathfrak{b}(\mathfrak{l}'_c), \mathfrak{h}'_c)$ as a positive root system of $\Delta(\mathfrak{l}'_c, \mathfrak{h}'_c)$. If $\lambda'_c \in (\mathfrak{h}'_c)^*$ is dominant integral for $\Delta(\mathfrak{b}(\mathfrak{l}'_c), \mathfrak{h}'_c)$, write $F(\lambda'_c)$ for the irreducible finite-dimensional representation of $\mathfrak{l}'_c$ with highest weight λ'_c.

Let Λ be the set consisting of $\lambda' = \lambda'_c + \lambda'_n \in \mathfrak{z}(\mathfrak{l}'')^*$ such that

- λ' is linear,
- λ'_c is dominant for $\Delta(\mathfrak{b}(\mathfrak{l}'_c), \mathfrak{h}'_c)$, and
- $\lambda'_n = 0$.

For $\lambda' \in \Lambda$, define the representation $F(\lambda')$ of $\mathfrak{l}' = \mathfrak{l}'_c \oplus \mathfrak{l}'_n$ by the exterior tensor product of $F(\lambda'_c)$ and the trivial representation of $\mathfrak{l}'_n$:

$$F(\lambda') := F(\lambda'_c) \boxtimes \mathbb{C}.$$

Since λ' is linear, $F(\lambda')$ lifts to a representation of L'. Define

(6.6)

$$m(\lambda', p) := \dim \operatorname{Hom}_{\overline{Q}\cap L'}\left(F(\lambda'), \mathbb{C}_\lambda|_{\overline{Q}\cap G'} \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/(\bar{\mathfrak{q}} + \mathfrak{g}')) \otimes S^p(\mathfrak{g}/(\bar{\mathfrak{q}} + \mathfrak{g}'))\right).$$

Theorem 6.4. *Let the notation and the assumption be as in Theorem 6.3. Define $\mathfrak{q}''$ as in (5.1) and define Λ, $m(\lambda', p)$ as above. Then there exists an injective homomorphism of $(\mathfrak{g}', K')$-modules*

$$A_{\mathfrak{q}}(\lambda) \to \bigoplus_{p=0}^{\infty} \bigoplus_{\lambda' \in \Lambda} A_{\mathfrak{q}''}(\lambda')^{\oplus m(\lambda', p)}. \tag{6.7}$$

Proof. We use the notation of the proof of Theorem 6.3. In light of (6.5), it is enough to show that

$$\Gamma(X', i'_{+}\mathcal{O}_{Y'} \otimes_{\mathcal{O}_{X'}} \mathcal{W}^p_{X'}) \simeq \bigoplus_{\lambda' \in \Lambda} A_{\mathfrak{q}''}(\lambda')^{\oplus m(\lambda', p)}. \tag{6.8}$$

Let us prove that

$$W^p \simeq \bigoplus_{\lambda' \in \Lambda} F(\lambda')^{\oplus m(\lambda', p)} \tag{6.9}$$

as L'-modules. Let F be an irreducible finite-dimensional L'-module such that $\mathrm{Hom}_{L'}(F, W^p) \neq 0$. Then the Frobenius reciprocity shows $\mathrm{Hom}_{\overline{Q}\cap L'}(F, \mathbb{C}_\lambda \otimes V^p) \neq 0$. Since L' is connected, F is irreducible as an $\mathfrak{l}'$-module. Hence the $\mathfrak{l}'$-module F is written as the exterior tensor product $F_c \boxtimes F_n$ for an irreducible $\mathfrak{l}'_c$-module F_c and an irreducible $\mathfrak{l}'_n$-module F_n. Since λ is linear and unitary, Remark 2.5 implies that $\bar{\mathfrak{q}} \cap \mathfrak{p}$ acts by zero on $\mathbb{C}_\lambda$. Hence $\mathfrak{l}'_n$ also acts by zero on $\mathbb{C}_\lambda$. Moreover, Lemma 5.7 implies that $\mathfrak{l}'_n$ acts by zero on $\mathfrak{g}/(\bar{\mathfrak{q}} + \mathfrak{g}')$. Therefore, $\mathfrak{l}'_n$ acts by zero on W^p. As a consequence, F_n must be the trivial representation and $F \simeq F(\lambda')$ for some $\lambda' \in \Lambda$. Then the Frobenius reciprocity gives

$$m(\lambda', p) := \dim \mathrm{Hom}_{\overline{Q}\cap L'}(F(\lambda'), \mathbb{C}_\lambda \otimes V^p) = \dim \mathrm{Hom}_{L'}(F(\lambda'), W^p),$$

and hence (6.9) is proved.

We set

$$X'' = G'/\overline{Q''}, \quad Y'' = K'/(\overline{Q''} \cap K'),$$

$$\begin{array}{ccc} Y'' & \xrightarrow{i''} & X'' \\ \downarrow & & \downarrow \pi' \\ Y' & \xrightarrow{i'} & X' \end{array}$$

where the maps are defined canonically. By the same argument as in the proof of Lemma 6.1, we can prove that this diagram is a Cartesian square. Take $\lambda' \in \Lambda$ and write $\mathcal{L}_{\lambda', X''}$ for the $\mathcal{O}_{X''}$-module associated with the $\overline{Q''}$-module $\mathbb{C}_{\lambda'}$. Theorem 4.1 shows that

$$A_{\mathfrak{q}''}(\lambda') \simeq \Gamma(X'', i''_{+}\mathcal{O}_{Y''} \otimes_{\mathcal{O}_{X''}} \mathcal{L}_{\lambda', X''}). \tag{6.10}$$

As in the proof of Theorem 6.3, we see that

$$\pi'_*(i''_{+}\mathcal{O}_{Y''} \otimes_{\mathcal{O}_{X''}} \mathcal{L}_{\lambda', X''}) \simeq i'_{+}\mathcal{O}_{Y'} \otimes_{\mathcal{O}_{X'}} \pi'_*(\mathcal{L}_{\lambda', X''}).$$

Put $S' := \overline{Q'}/\overline{Q''}$ and write $\mathcal{L}_{\lambda', S'}$ for the $\mathcal{O}_{S'}$-module associated with $\mathbb{C}_{\lambda'}$. The decompositions

$$\overline{\mathfrak{q}'} = \mathfrak{l}'_c \oplus \mathfrak{l}'_n \oplus \overline{\mathfrak{u}'}, \quad \overline{\mathfrak{q}''} = \mathfrak{b}(\mathfrak{l}'_c) \oplus \mathfrak{l}'_n \oplus \overline{\mathfrak{u}'}$$

show that S' is isomorphic to the complete flag variety of the reductive Lie algebra $\mathfrak{l}'_c$. Hence by the Borel–Weil theorem, $\Gamma(S', \mathcal{L}_{\lambda', S'}) \simeq F(\lambda')$. Then it follows from Lemma 3.2 that

$$\pi'_*(\mathcal{L}_{\lambda', X''}) \simeq \mathcal{F}(\lambda')_{X'},$$

where $\mathcal{F}(\lambda')_{X'}$ is the $\mathcal{O}_{X'}$-module associated with the $\overline{Q'}$-module $F(\lambda')$. As a consequence, we have

$$\begin{aligned}\Gamma(X'', i''_+\mathcal{O}_{Y''} \otimes_{\mathcal{O}_{X''}} \mathcal{L}_{\lambda',X''}) &\simeq \Gamma(X', \pi'_*(i''_+\mathcal{O}_{Y''} \otimes_{\mathcal{O}_{X''}} \mathcal{L}_{\lambda',X''})) \\ &\simeq \Gamma(X', i'_+\mathcal{O}_{Y'} \otimes_{\mathcal{O}_{X'}} \mathcal{F}(\lambda')_{X'}).\end{aligned} \tag{6.11}$$

The isomorphism (6.8) follows from (6.9), (6.10), and (6.11). □

Remark 6.5. On the right side of (6.7), λ' may not be in the weakly fair range even if $m(\lambda', p) > 0$.

7. Associated Varieties

As a corollary to Theorem 6.4, we determine the associated variety of $(\mathfrak{g}', K')$-modules that occur in $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}',K')}$.

For a finitely generated $\mathfrak{g}$-module V, write $\mathrm{Ass}_{\mathfrak{g}}(V)$ for the associated variety of V. We use the following fact on associated varieties.

Fact 7.1 ([12])**.** *Let $\mathfrak{g}$ be a complex reductive Lie algebra.*

(1) $\mathrm{Ass}_{\mathfrak{g}}(V) = \mathrm{Ass}_{\mathfrak{g}}(V \otimes F)$ *for any finitely generated $\mathfrak{g}$-module V and a nonzero finite-dimensional representation F of $\mathfrak{g}$.*

(2) *If λ is in the weakly fair range and $A_{\mathfrak{q}}(\lambda)$ is nonzero, then* $\mathrm{Ass}_{\mathfrak{g}}(A_{\mathfrak{q}}(\lambda)) = \mathrm{Ad}(K)(\bar{\mathfrak{u}} \cap \mathfrak{p})$. *Here, we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by a non-degenerate invariant bilinear form.*

Fact 7.1 (2) can be generalized in the following way.

Proposition 7.2. *Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ and $\mathbb{C}_\lambda$ a one-dimensional $(\mathfrak{l}, L \cap K)$-module. Suppose that V is an irreducible $(\mathfrak{g}, K)$-submodule of $A_{\mathfrak{q}}(\lambda)$. Then* $\mathrm{Ass}_{\mathfrak{g}}(V) = \mathrm{Ad}(K)(\bar{\mathfrak{u}} \cap \mathfrak{p})$.

Proof. If we take sufficiently large integer $N \in \mathbb{N}$, then $\lambda + 2N\rho(\mathfrak{u})$ is in the good range. In view of Fact 7.1 (2), it is enough to show that $\mathrm{Ass}_{\mathfrak{g}}(V) = \mathrm{Ass}_{\mathfrak{g}}(A_{\mathfrak{q}}(\lambda + 2N\rho(\mathfrak{u})))$. Let F be the irreducible finite-dimensional $(\mathfrak{g}, K)$-module with lowest weight $-2N\rho(\mathfrak{u})$. Then there is an injective $(\bar{\mathfrak{q}}, L \cap K)$-homomorphism $\mathbb{C}_\lambda \to F \otimes \mathbb{C}_{\lambda+2N\rho(\mathfrak{u})}$, which gives a long exact sequence:

$$\begin{aligned} &\cdots \to (P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_{s+1}((F \otimes \mathbb{C}_{\lambda+2N\rho(\mathfrak{u})})/\mathbb{C}_\lambda) \to \\ &\to (P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_s(\mathbb{C}_\lambda) \to (P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_s(F \otimes \mathbb{C}_{\lambda+2N\rho(\mathfrak{u})}) \to \cdots .\end{aligned}$$

We claim that $(P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_{s+1}((F \otimes \mathbb{C}_{\lambda+2N\rho(\mathfrak{u})})/\mathbb{C}_\lambda) = 0$. Indeed, $(F \otimes \mathbb{C}_{\lambda+2N\rho(\mathfrak{u})})/\mathbb{C}_\lambda$ admits a finite filtration $\{F_p\}$ of $(\bar{\mathfrak{q}}, L \cap K)$-modules such that $\bar{\mathfrak{u}}$ acts by zero on F_p/F_{p-1}. Then [8, Theorem 5.35] shows that $(P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_{s+1}(F_p/F_{p-1}) = 0$. By using the exact sequences

$$(P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_{s+1}(F_{p-1}) \to (P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_{s+1}(F_p) \to (P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_{s+1}(F_p/F_{p-1})$$

iteratively, we can see that $(P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_{s+1}((F \otimes \mathbb{C}_{\lambda+2N\rho(\mathfrak{u})})/\mathbb{C}_\lambda) = 0$.

As a result, we get an injective map

$$V \subset A_{\mathfrak{q}}(\lambda) \to (P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K})_s(F \otimes \mathbb{C}_{\lambda+2N\rho(\mathfrak{u})}) \simeq F \otimes A_{\mathfrak{q}}(\lambda + 2N\rho(\mathfrak{u})),$$

where the last isomorphism is the Mackey isomorphism [8, Theorem 2.103]. Then Fact 7.1 (1) shows that

$$\mathrm{Ass}_{\mathfrak{g}}(V) \subset \mathrm{Ass}_{\mathfrak{g}}\big(F \otimes A_{\mathfrak{q}}(\lambda + 2N\rho(\mathfrak{u}))\big) = \mathrm{Ass}_{\mathfrak{g}}\big(A_{\mathfrak{q}}(\lambda + 2N\rho(\mathfrak{u}))\big).$$

For the opposite inclusion, we see that

$$\mathrm{Hom}_{\mathfrak{g},K}\big(V \otimes F^*, A_{\mathfrak{q}}(\lambda + 2N\rho(\mathfrak{u}))\big) \simeq \mathrm{Hom}_{\mathfrak{g},K}\big(V, F \otimes A_{\mathfrak{q}}(\lambda + 2N\rho(\mathfrak{u}))\big) \neq 0.$$

Since $A_{\mathfrak{q}}(\lambda+2N\rho(\mathfrak{u}))$ is irreducible, there exists a surjective map $V \otimes F^* \to A_{\mathfrak{q}}(\lambda+2N\rho(\mathfrak{u}))$. Therefore, Fact 7.1 (1) shows that

$$\mathrm{Ass}_{\mathfrak{g}}(V) = \mathrm{Ass}_{\mathfrak{g}}(V \otimes F^*) \supset \mathrm{Ass}_{\mathfrak{g}}\big(A_{\mathfrak{q}}(\lambda + 2N\rho(\mathfrak{u}))\big).$$

Consequently,

$$\mathrm{Ass}_{\mathfrak{g}}(V) = \mathrm{Ass}_{\mathfrak{g}}\big(A_{\mathfrak{q}}(\lambda + 2N\rho(\mathfrak{u}))\big) = \mathrm{Ad}(K)(\bar{\mathfrak{u}} \cap \mathfrak{p}).$$

□

Remark 7.3. In some literature, $A_{\mathfrak{q}}(\lambda)$ is defined by using the derived functor of $I^{\mathfrak{g},K}_{\mathfrak{q},L\cap K}$. If we adopt this definition, we have to replace 'irreducible $(\mathfrak{g}, K)$-submodule' in Proposition 7.2 by 'irreducible quotient $(\mathfrak{g}, K)$-module'. Both definitions agree if λ is unitary and in the weakly fair range.

A connection between branching laws of $\mathfrak{g}$-modules and their associated varieties was studied in [12].

Fact 7.4 ([12, Theorem 3.1])**.** *Let $\mathfrak{h}$ be a reductive Lie subalgebra of $\mathfrak{g}$. Write $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}} : \mathfrak{g}^* \to \mathfrak{h}^*$ for the restriction map. Suppose that W is an irreducible $\mathfrak{g}$-module and V is an irreducible $\mathfrak{h}$-module such that $\mathrm{Hom}_{\mathfrak{h}}(V, W) \neq 0$. Then*

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\mathrm{Ass}_{\mathfrak{g}}(W)) \subset \mathrm{Ass}_{\mathfrak{h}}(V).$$

In our setting, we can deduce from Theorem 6.4 that the equality holds.

Theorem 7.5. *Let the notation and the assumption be as in Theorem 6.3. Suppose that V is an irreducible $(\mathfrak{g}', K')$-module such that $\mathrm{Hom}_{\mathfrak{g}'}(V, A_{\mathfrak{q}}(\lambda)) \neq 0$. Then*

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathrm{Ass}_{\mathfrak{g}}(A_{\mathfrak{q}}(\lambda))) = \mathrm{Ass}_{\mathfrak{g}'}(V).$$

Proof. In light of Theorem 6.4, we see that V is isomorphic to an irreducible $(\mathfrak{g}', K')$-submodule of $A_{\mathfrak{q}''}(\lambda')$ for some character λ'. Then Proposition 7.2 and Fact 7.1 (2) show that

$$\mathrm{Ass}_{\mathfrak{g}'}(V) = \mathrm{Ad}(K')(\overline{\mathfrak{u}''} \cap \mathfrak{p}'), \quad \mathrm{Ass}_{\mathfrak{g}}(A_{\mathfrak{q}}(\lambda)) = \mathrm{Ad}(K)(\overline{\mathfrak{u}} \cap \mathfrak{p}).$$

Therefore, it is enough to prove that

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathrm{Ad}(K)(\mathfrak{u} \cap \mathfrak{p})) = \mathrm{Ad}(K')(\mathfrak{u}'' \cap \mathfrak{p}').$$

Since $\mathfrak{q}$ is σ-open, $K'/(Q \cap K')$ is open dense in the partial flag variety $K/(Q \cap K)$. As a result, $\mathrm{Ad}(K')(\mathfrak{u}\cap\mathfrak{p})$ is dense in $\mathrm{Ad}(K)(\mathfrak{u}\cap\mathfrak{p})$ and hence $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathrm{Ad}(K')(\mathfrak{u}\cap\mathfrak{p}))$ is dense in $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathrm{Ad}(K)(\mathfrak{u} \cap \mathfrak{p}))$. From the proof of Proposition 5.4, we have

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathfrak{u} \cap \mathfrak{p}) = \mathfrak{u}' \cap \mathfrak{p}' = \mathfrak{u}'' \cap \mathfrak{p}'.$$

Consequently, $\mathrm{Ad}(K')(\mathfrak{u}'' \cap \mathfrak{p}')$ is a dense subset of $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathrm{Ad}(K)(\mathfrak{u} \cap \mathfrak{p}))$. Since $\mathrm{Ad}(K')(\mathfrak{u}'' \cap \mathfrak{p}')$ is closed, we conclude that

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathrm{Ad}(K)(\mathfrak{u} \cap \mathfrak{p})) = \mathrm{Ad}(K')(\mathfrak{u}'' \cap \mathfrak{p}'),$$

which completes the proof. □

References

[1] F. Bien, "$\mathcal{D}$-Modules and Spherical Representations", Math. Notes 39, Princeton U.P., 1990.

[2] A. Borel, "Linear Algebraic Groups", Second edition. Graduate Texts in Mathematics, 126. Springer-Verlag, 1991.

[3] M. Duflo, J. Vargas, *Branching laws for square integrable representations*, Proc. Japan Acad. Ser. A Math. Sci. **86** (2010), 49–54.

[4] B. Gross, N. Wallach, *Restriction of small discrete series representations to symmetric subgroups*, Proc. Sympos. Pure Math. **68** (2000), 255–272.

[5] H. Hecht, D. Miličić, W. Schmid, J. A. Wolf, *Localization and standard modules for real semisimple Lie groups. I. The duality theorem*, Invent. Math. **90** (1987), 297–332.

[6] G. Hochschild, "Basic Theory of Algebraic Groups and Lie Algebras", Graduate Texts in Mathematics, 75. Springer-Verlag, 1981.

[7] S. Kitchen, *Localization of cohomologically induced modules to partial flag varieties*, Ph.D. Thesis, The University of Utah (2010).

[8] A. W. Knapp, D. Vogan, "Cohomological Induction and Unitary Representations", Princeton U.P., 1995.

[9] T. Kobayashi, *The restriction of $A_{\mathfrak{q}}(\lambda)$ to reductive subgroups*, Proc. Japan Acad. Ser. A Math. Sci. **69** (1993), 262–267.

[10] T. Kobayashi, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications*, Invent. Math. **117** (1994), 181–205.

[11] T. Kobayashi, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups. II. —micro-local analysis and asymptotic K-support*, Ann. of Math. **147** (1998), 709–729.

[12] T. Kobayashi, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups. III. —restriction of Harish-Chandra modules and associated varieties*, Invent. Math. **131** (1998), 229–256.

[13] T. Kobayashi, *Branching problems of Zuckerman derived functor modules*, to appear in Contemp. Math., In: Representation Theory and Mathematical Physics in honor of Gregg Zuckerman (J. Adams, B. Lian, S. Sahi, eds.), 27pp. arXiv:1104.4399 (2011).

[14] T. Kobayashi, B. Ørsted, *Analysis on the minimal representation of $O(p,q)$. II. Branching laws*, Adv. Math. **180** (2003), 513-550.

[15] T. Kobayashi, Y. Oshima, *Classification of discretely decomposable $A_{\mathfrak{q}}(\lambda)$ with respect to reductive symmetric pairs*, 31pp. arXiv:1104.4400 (2011).

[16] H. Y. Loke, *Restrictions of quaternionic representations*, J. Funct. Anal. **172** (2000), 377–403.

[17] D. Miličić, P. Pandžić, *Equivariant derived categories, Zuckerman functors and localization*, Geometry and representation theory of real and p-adic groups, (J. Tirao, D. Vogan, J. A. Wolf, eds.), Prog. Math. **158**, Birkhäuser (1996), 209–242.

[18] D. Mumford, "Geometric Invariant Theory", Springer-Verlag, 1965.

[19] B. Ørsted, B. Speh, *Branching laws for some unitary representations of $SL(4,\mathbb{R})$*, SIGMA Symmetry Integrability Geom. Methods Appl., **4** (2008), 19pp.

[20] H. Sekiguchi, *Branching rules of Dolbeault cohomology groups over indefinite Grassmannian manifolds*, Proc. Japan Acad. Ser. A Math. Sci. **87** (2011), 31–34.

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